AD-A176 504 PERFORMANCE OF DIGITAL COMMUNICATION RECEIVERS OPERATING IN THE PRESENCE OF WHITE GAUSSIAN NONSTATIONARY NOISE(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA Y J KIM DEC 86 1/1

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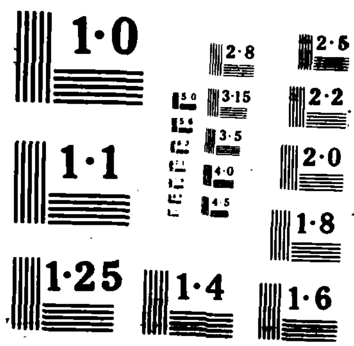
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

PERFORMANCE OF DIGITAL COMMUNICATION
RECEIVERS OPERATING
IN THE PRESENCE OF
WHITE GAUSSIAN NONSTATIONARY NOISE

by

Young Joo Kim

December 1986

Thesis Advisor Daniel C. Bukofzer

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION: Unclassified

1b RESTRICTIVE MARKINGS:

2a SECURITY CLASSIFICATION AUTHORITY:

2b DECLASSIFICATION / DOWNGRADING SCHEDULE:

3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution is unlimited.

4 PERFORMING ORGANIZATION REPORT NUMBER(S):

5 MONITORING ORGANIZATION REPORT NUMBER(S):

6a NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School

6b OFFICE SYMBOL (If applicable): 62

7a NAME OF MONITORING ORGANIZATION: Naval Postgraduate School

6c ADDRESS (City, State, and ZIP Code): Monterey, California 93943-5000

7b ADDRESS (City, State, and ZIP Code): Monterey, California 93943-5000

8a NAME OF FUNDING/SPONSORING ORGANIZATION:

8b OFFICE SYMBOL (If applicable):

9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER:

8c ADDRESS (City, State, and ZIP Code):

10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification): PERFORMANCE OF DIGITAL COMMUNICATION RECEIVERS OPERATING IN THE PRESENCE OF WHITE GAUSSIAN NONSTATIONARY NOISE

12 PERSONAL AUTHOR(S): Young Joo Kim

13a TYPE OF REPORT: Master's thesis

13b TIME COVERED: FROM ______ TO ______

14 DATE OF REPORT (Year, Month, Day): 1986 December

15 PAGE COUNT: 76

16 SUPPLEMENTARY NOTATION:

17 COSATI CODES

FIELD	GROUP	SUB-GROUP

18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number): AWGN, Nonstationary AWGN, Suboptimum Receiver, Optimum Receiver, JSR·SNR

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

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20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS

21 ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a NAME OF RESPONSIBLE INDIVIDUAL: Daniel C. Bukofzer

22b TELEPHONE (Include Area Code): 408-646-2849

22c OFFICE SYMBOL: 62

DD FORM 1473, 84 MAR

83 APR edition may be used until exhausted
All other editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE

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Performance of Digital Communication Receivers Operating in the Presence of White Gaussian Nonstationary Noise

by

Young Joo Kim
Major, Korea Army
B.S., Korea Military Academy, 1975

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
December 1986

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ABSTRACT

The problem of evaluating the performance of digital communication receivers operating in the presence of additive white Gaussian noise (AWGN) and nonstationary AWGN is addressed. A specific model for the nonstationary AWGN is proposed and the corresponding performance of conditional digital communication receivers is derived. Additionally, receivers that are optimum (in minimum probability of error sense) for detecting binary signals in the presence of noise and the nonstationary interference modeled is derived and its performance evaluated. Several examples involving Phase Reversal Keyed modulation and Frequency Shift keyed modulation for various forms of the nonstationary interference are worked out.

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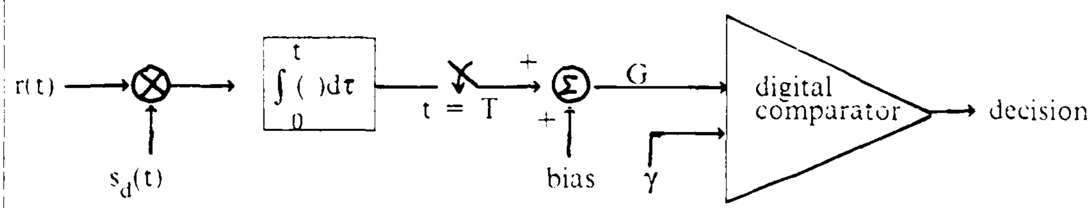
ACKNOWLEGEMEMTS

I would like to express my deep appreciation to Professor Daniel C. Bukofzer for his patience and assistance in completing this thesis. Thanks also to Professor Glen A. Myers for his kindly assistance. Special thanks goes to my wife Shin, Hyun Soon and son Soo Min and daughters Yoon Yi and Soo Jin for support and understanding during my stay at U.S. Naval Postgraduate School.

I. INTRODUCTION

The theory of signal detection in additive white Gaussian noise (WGN) is well established. Optimum structures and their performance can be found in many textbooks [Ref. 1: Chap. 4]. Solution of the so-called Binary Hypothesis Testing problem allows one to obtain an optimum structure (receiver) that is capable of deciding with minimum probability of error, which of two possible signals, $s_1(t)$ or $s_0(t)$, was transmitted over an additive white Gaussian noise (AWGN) channel. Such a receiver is shown in Figure 1.1



$$r(t) = \begin{cases} s_1(t) + \mathrm{WGN} & \text{w.p. } p_1 \\ s_0(t) + \mathrm{WGN} & \text{w.p. } p_0 \end{cases} \qquad \gamma = \frac{N_0}{2} \ln \lambda_0$$

$$\text{bias} = \frac{1}{2} \int_T [\, s_0^2(t) - s_1^2(t) \,]\, dt \qquad s_d(t) = s_1(t) - s_0(t)$$

Figure 1.1 Structure of the Optimum Receiver.

and its performance (i.e., probability of error, P_e) is given by [Ref. 2: Chap. 6],

$$P_e = \frac{p_0}{2} \operatorname{erfc}\left[\frac{\gamma + E(1-\rho)}{\sqrt{2N_0 E(1-\rho)}} \right] + \frac{p_1}{2} \operatorname{erf}\left[\frac{\gamma - E(1-\rho)}{\sqrt{2N_0 E(1-\rho)}} \right] \tag{1.1}$$

where

p_i = Probability that $s_i(t)$ was transmitted, $i = 0,1$

$N_0/2$ = Power spectral density (PSD) level of the additive WGN

$$E = \frac{1}{2}\int_T [\, s_0^2(t) + s_1^2(t)\,]\, dt \tag{1.2}$$

T = Length of the observation interval over which signals are received

$$\rho = \frac{1}{E}\int_T s_0(t)s_1(t)\, dt \tag{1.3}$$

$$\gamma = \text{Threshold} = \frac{N_0}{2}\,\ell n\, \lambda_0. \tag{1.4}$$

The erfc(x) and erf(x) functions are defined by

$$\mathrm{erfc}(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}\, d\xi$$

$$\mathrm{erf}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}\, d\xi$$

and λ_0 depends on p_0 and p_1 primarily, but may also depend on costs associated with each decision (whether correct or incorrect). If these costs are included but are all assumed to be equal, and $p_0 = p_1$, then $\lambda_0 = 1$.

The design of the receiver shown in Figure 1.1 does not take into account the possibility that in the transmission channel, in addition to the AWGN, another source of interference (perhaps intentional, such as a jammer) may be present, with characteristics far different from the assumed AWGN interference.

In this thesis, we analyze the performance of the receiver of Figure 1.1, when in addition to the AWGN, an additive nonstationary Gaussian noise is present in the

channel. In Chapter II, this nonstationary Gaussian noise component is described, and its possible generation by an intentional interferer is analyzed. In Chapter III, the performance of the receiver of Figure 1.1 is analyzed taking into account the presence of the nonstationary interference. It is demonstrated that the resulting performance is in all cases worse (i.e., higher P_e) than that predicted by Equation 1.1. (This is to be expected, since the model that gave rise to Equation 1.1 is not optimum for any interference other than the AWGN). Chapter IV presents a modification to the receiver of Figure 1.1 that results in a structure that is optimum for the detection of binary signals in AWGN and additive nonstationary interference of the type modeled in Chapter II. The performance of this modified structure is evaluated and compared with the results of Chapter III. In Chapter V several examples are presented for two important types of signaling schemes, namely Phase Reversal Keying (PRK) and Frequency Shift Keying (FSK) under various assumptions about the characteristics of the interference. The derived mathematical results are used to evaluate receiver performance in terms of probability of error, P_e, as a function of signal to noise ratio, and interference to signal ratio. The results are then interpreted and conclusions are drawn.

II. NONSTATIONARY INTERFERENCE MODEL

As pointed out in Chapter I, the intentional interference $n_j(t)$ is modeled as a nonstationary Gaussian process having autocorrelation function $R_{n_j}(t,\tau)$. A possible method for generating such a process is depicted in Figure 2.1, where a Gaussian process $n_g(t)$ is multiplied by the deterministic function $q(t)$ to produce the process $n_j(t)$.

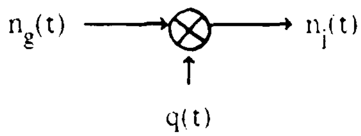


Figure 2.1 Structure for the Generation of $n_j(t)$.

That is

$$n_j(t) = q(t)n_g(t)$$

so that

$$R_{n_j}(t,\tau) = E\{ n_j(t)n_j(\tau) \} = q(t)q(\tau)R_{n_g}(t,\tau)$$

where

$$R_{n_g}(t,\tau) = E\{ n_g(t)n_g(\tau) \}.$$

Note that $n_j(t)$ is a Gaussian process since $n_g(t)$ is assumed to be Gaussian. If $n_g(t)$ is white with unit power spectral density level, then $R_{n_g}(t,\tau) = \delta(t-\tau)$ so that

$$R_{n_j}(t,\tau) = q(t)q(\tau)\delta(t-\tau) = q^2(t)\delta(t-\tau) = Q(t)\delta(t-\tau) \tag{2.1}$$

where

$$Q(t) \triangleq q^2(t) \geq 0.$$

If $n_g(t)$ is a nonwhite process, then

$$E\{n_j(t)n_j(t+\tau)\} = R_{n_j}(t,t+\tau) = q(t)q(t+\tau)R_{n_g}(\tau). \tag{2.2}$$

Implicit in these expressions is the assumed wide sense stationarity of $n_g(t)$. Observe from Equation 2.1 and Equation 2.2, that $n_j(t)$ is a white (colored) nonstationary Gaussian process when $n_g(t)$ is a white (colored) Gaussian process.

Since

$$\langle R_{n_j}(t,t+\tau)\rangle = \mathscr{R}_{n_j}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} q(t)q(t+\tau)R_{n_g}(\tau)\,dt$$

$$= Q_a(\tau)R_{n_g}(\tau)$$

where

$$Q_a(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} q(t)q(t+\tau)\,dt, \tag{2.3}$$

the average power of $n_j(t)$ is given by

$$P_{n_j} \equiv \mathscr{R}_{n_j}(0) = Q_a(0)R_{n_g}(0). \tag{2.4}$$

This expression has physical meaning only when $R_{n_g}(0)$ is finite (i.e., $n_g(t)$ is non-white). Note that when $n_g(t)$ is white (with unit PSD level), $Q_a(0)$ becomes the average PSD level of $n_j(t)$.

For most the work in this thesis, it will be assumed that the interference $n_j(t)$ is a white nonstationary Gaussian process, with autocorrelation given by Equation 2.1. Several forms of $q(t)$ used in the examples worked out in Chapter V are shown in Figure 2.2.

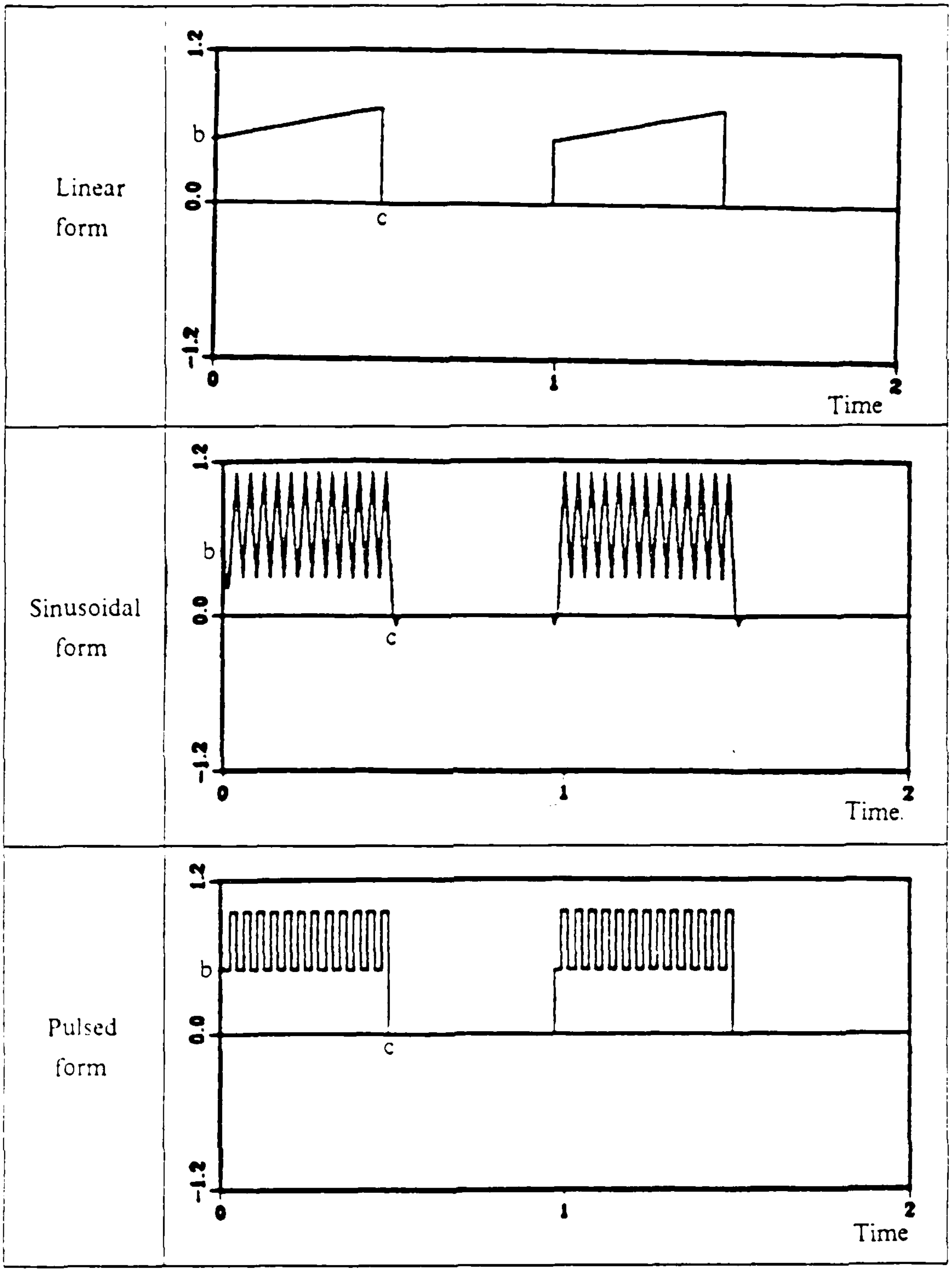


Figure 2.2 Several Examples of q(t) as a Function of Normalized Time.

III. RECEIVER PERFORMANCE ANALYSIS

We now evaluate the performance of the receiver depicted in Figure 1.1 when

$$r(t) = \begin{cases} s_1(t) + n_w(t) + n_j(t) & \text{w.p. } p_1 \\ s_0(t) + n_w(t) + n_j(t) & \text{w.p. } p_0 \end{cases} \qquad t \in T, \tag{3.1}$$

where $n_w(t)$ represents the AWGN with PSD level $N_0/2$, and $n_j(t)$ is the white nonstationary Gaussian noise with autocorrelation function given by Equation 2.1. Observe from Equation 3.1 that $r(t)$ is a Gaussian random process, and its processing by the receiver of Figure 1.1 will result in G being a Gaussian random variable. Observe from Figure 1.1 that the receiver performs the test

$$G = \int_T r(t)s_d(t)\,dt + \frac{1}{2}\int_T [\, s_0^2(t) - s_1^2(t)\,] \quad \gtrless \quad \gamma$$

and if $G > \gamma$ the presence of $s_1(t)$ is declared, whereas if $G < \gamma$, the presence of $s_0(t)$ is declared. Whether $s_1(t)$ or $s_0(t)$ was transmitted, G is a Gaussian random variable having conditional means

$$m_1 \triangleq E\{G \mid s_1(t) \text{ was transmitted}\} = \frac{1}{2}\int_T [\, s_1(t) - s_0(t)\,]^2\,dt \tag{3.2}$$

$$m_0 \triangleq E\{G \mid s_0(t) \text{ was transmitted}\} = -\frac{1}{2}\int_T [\, s_1(t) - s_0(t)\,]^2\,dt = -m_1 \tag{3.3}$$

and conditional variances

$$\text{var}\{G \mid s_i(t) \text{ was transmitted}\} = E\{\, [\int_T [\, n_w(t) + n_j(t)\,]\, s_d(t)\,dt\,]^2 \,\}$$

$$= \int_T [\, \frac{N_0}{2} + Q(t)\,]\, s_d^2(t)\,dt = \sigma_G^2 \qquad i = 0,1 \tag{3.4}$$

where we have assumed that $n_w(t)$ and $n_j(t)$ are uncorrelated zero mean random process. Let

$f_{G_i}(g_i)$ = probability density function of G conditioned on $s_i(t)$ being transmitted, $i = 0,1$

so that

$$
\begin{aligned}
P_{e,s} &= P\{\text{decide } s_1(t) \text{ transmitted} \mid s_0(t) \text{ was transmitted}\}\ p_0 \\
&\quad + P\{\text{decide } s_0(t) \text{ transmitted} \mid s_1(t) \text{ was transmitted}\}\ p_1 \\
&= p_0 \int_{\gamma}^{\infty} f_{G_0}(g_0)\, dg_0 + p_1 \int_{-\infty}^{\gamma} f_{G_1}(g_1)\, dg_1
\end{aligned}
$$

(the subscript s denotes that receiver is suboptimum for the noise model used), and from Equations (3.2) - (3.4)

$$P_{e,s} = \frac{p_0}{2} \operatorname{erfc}[\, (\gamma - m_0)/\sqrt{2}\, \sigma_G \,] + \frac{p_1}{2} \operatorname{erf}[\, (\gamma - m_1)/\sqrt{2}\, \sigma_G \,]. \tag{3.5}$$

Comparison of Equation 1.1 with Equation 3.5 is best achieved if we assume equal decision costs and equal prior probabilities p_0 and p_1, so that $\gamma = 0$ (Under these assumptions $\lambda_0 = 1$ which results in γ becoming 0). Thus from Equations (1.1) - (1.4)

$$P_e = \operatorname{erfc}[\, \sqrt{E(1 - \rho)/2N_0}\,] \tag{3.6}$$

(note that $E(1 - \rho) \geq 0$) and from Equation 3.5

$$P_{e,s} = \operatorname{erfc}(\, m_1/\sqrt{2}\, \sigma_G \,).$$

From Equation 3.2 and Equation 3.4 we obtain

$$\frac{m_1^2}{\sigma_G^2} = \Big[\frac{1}{2}\int_T s_d^2(t)\,dt\Big]^2 \Big/ \Big[\frac{N_0}{2}\int_T s_d^2(t)\,dt + \int_T Q(t)s_d^2(t)\,dt\Big]$$

$$= \frac{E(1-\rho)}{N_0}\Big[1 - \frac{\int_T Q(t)s_d^2(t)\,dt}{N_0E(1-\rho) + \int_T Q(t)s_d^2(t)\,dt}\Big]$$

and therefore

$$P_{e,s} = \operatorname{erfc}\Big[\sqrt{E(1-\rho)(1-C)/2N_0}\Big] \tag{3.7}$$

where

$$C = \int_T Q(t)s_d^2(t)\,dt \Big/ \Big[N_0E(1-\rho) + \int_T Q(t)s_d^2(t)\,dt\Big]. \tag{3.8}$$

Observe that $C \in [0,1]$ and therefore is the factor that causes an increase in probability of error due to the presence of $n_j(t)$. If $n_j(t)$ were not present, then $Q(t) \equiv 0 \rightarrow C = 0$ and Equation 3.7 would become identical to Equation 3.6. Thus we can study the performance degradation of the receiver by either evaluating $P_{e,s}$, or by evaluating C, as its size will dictate the increase in $P_{e,s}$. Note that if $Q(t) \rightarrow \infty$, then $C \rightarrow 1$ and $P_{e,s} \rightarrow 1/2$ as expected. It is worthwhile noting that if the nonstationary interference become (white) stationary so as to simply have the effect of raising the AWGN level, then, with $q(t) = K$ (a constant), from Equation 3.8

$$C = \frac{K^2\int_T s_d^2(t)\,dt}{\frac{N_0}{2}\int_T s_d^2(t)\,dt + K^2\int_T s_d^2(t)\,dt} = \frac{K^2\Big/\frac{N_0}{2}}{1 + K^2\Big/\frac{N_0}{2}}. \tag{3.9}$$

Since the interference source will typically be an intentional jammer (in this case the jammer lacks sophistication), we can think of the $K^2/(N_0/2)$ ratio as

$$\frac{K^2}{N_0/2} = \frac{K^2E}{EN_0/2} = \left(\frac{E}{N_0/2}\right)\left(\frac{K^2}{E}\right) = (\text{signal to noise ratio})(\text{jammer to signal ratio})$$

and thus

$$P_{e,s} = \operatorname{erfc}\left[\sqrt{\frac{E(1-\rho)}{2N_0}\cdot\frac{1}{1+JSR\cdot SNR}}\right] \tag{3.10}$$

and

$$C = JSR\cdot SNR/(1 + JSR\cdot SNR)$$

where

$$SNR = E/(N_0/2)$$

$$JSR = K^2/E.$$

We see that a large JSR value is required for the system to become useless as a receiver (namely $P_{e,s} \to 1/2$). It therefore appears that a jammer can use its available power more efficiently by not spreading its power over large bandwidths. Consequently, jammer waveforms other than broadband noise power will be investigated in Chapter V. Note furthermore that an increasing value of N_0 in order to prevent JSR·SNR from getting too large is self defeating because as revealed by Equation 3.10, with JSR constant, $P_{e,s} \to 1/2$ as $N_0 \to \infty$.

The derived equation for $P_{e,s}$ will be used in the sequel to analyze specific modulation schemes and choices of nonstationary waveform $q(t)$. It will be demonstrated that by proper choice of $q(t)$, the receiver of Figure 1.1 can be rendered ineffective without the use of a great deal of jammer power.

IV. RECEIVER PERFORMANCE ANALYSIS WITH INTERFERENCE STATIONARIZATION

The optimum receiver for processing r(t) (with r(t) as given in Equation 3.1) is easily determined by realizing that the nonstationary interference can be stationarized by use of the (time-varying) system depicted in Figure 4.1

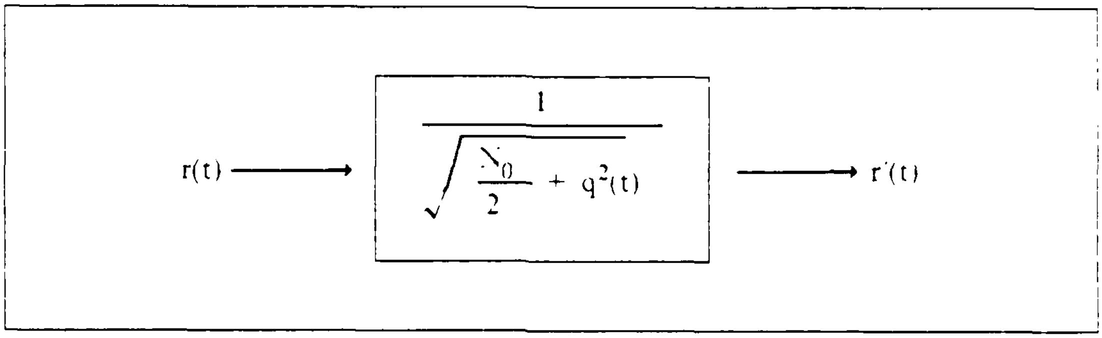


Figure 4.1 Structure for Stationarization.

That is, with

$$r(t) = s_i(t) + n_w(t) + n_j(t) \qquad i = 0,1$$

where $n_w(t)$ and $n_j(t)$ are described in the discussion following Equation 3.1, the output of the linear system depicted in Figure 4.1 is

$$r'(t) = s_i'(t) + n'(t) \qquad i = 0,1$$

where

$$s_i'(t) = \frac{s_i(t)}{\sqrt{\frac{N_0}{2} + q^2(t)}} \qquad i = 0,1$$

$$n'(t) = \frac{n_w(t) + n_j(t)}{\sqrt{\frac{N_0}{2} + q^2(t)}}. \tag{4.1}$$

Evaluating now the autocorrelation function of $n'(t)$, we obtain

$$E\{n'(t)n'(\tau)\} = E\{ \frac{[\, n_w(t) + n_j(t)\,]}{\sqrt{\frac{N_0}{2} + q^2(t)}} \cdot \frac{[\, n_w(\tau) + n_j(\tau)\,]}{\sqrt{\frac{N_0}{2} + q^2(\tau)}} \}$$

$$= \frac{R_{n_w}(t,\tau) + R_{n_j}(t,\tau)}{\sqrt{\frac{N_0}{2} + q^2(t)}\ \sqrt{\frac{N_0}{2} + q^2(\tau)}} \tag{4.2}$$

where the uncorrelatedness of $n_w(t)$ and $n_j(t)$ has been used in Equation 4.2. From Equation 2.1, we obtain

$$R_{n'}(t,\tau) = \frac{\frac{N_0}{2}\delta(t - \tau) + q^2(t)\delta(t - \tau)}{\sqrt{\frac{N_0}{2} + q^2(t)}\ \sqrt{\frac{N_0}{2} + q^2(t)}} = \delta(t-\tau)$$

and clearly, the output of the linear time varying system consists of a known signal $s_i'(t)$, $i=0,1$, in AWGN of unit PSD level. The optimum receiver for deciding whether $s_1'(t)$ or $s_0'(t)$ was received in AWGN of unit PSD level is given by the receiver of Figure 1.1, with $r(t)$ replaced by $r'(t)$, $s_d(t)$ replaced by

$$s_d'(t) = s_1'(t) - s_0'(t)$$

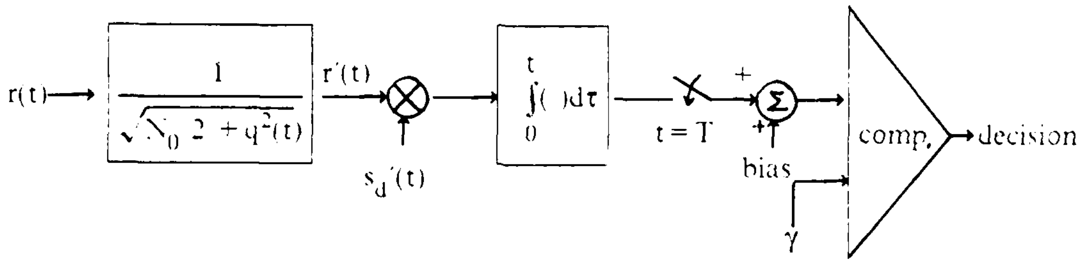
the bias term given by

$$\frac{1}{2}\int_T [\, s_0'^2(t) - s_1'^2(t)\,]\, dt$$

and

$$\gamma = \ln \lambda_0.$$

The structure of this optimum receiver is shown in Figure 4.2.



$$s_d'(t) = s_1'(t) - s_0'(t) \; ; \; s_i'(t) = \frac{s_i(t)}{\sqrt{\frac{N_0}{2} + q^2(t)}} \qquad i = 0,1$$

Figure 4.2 Structure of the Optimum Receiver with Nonstationary Interference.

Its performance is given by Equation 1.1 with

$$\gamma = \ell n \, \lambda_0$$

E replaced by E', where

$$E' = \frac{1}{2} \int_T [\, s_0'^2(t) + s_1'^2(t) \,] \, dt = \frac{1}{2} \int_T \frac{s_0^2(t) + s_1^2(t)}{\frac{N_0}{2} + q^2(t)} \, dt, \tag{4.3}$$

ρ replaced by ρ', where

$$\rho' = \frac{1}{E'} \int_T [\, s_0'(t) \, s_1'(t) \,] \, dt = \frac{1}{E'} \int_T \frac{s_0(t) s_1(t)}{\frac{N_0}{2} + q^2(t)} \, dt \tag{4.4}$$

and N_0 is replaced by 2 (since $N_0/2 = 1$ in this case). Assuming equal prior probabilities and decision costs, we can obtain from Equation 3.6, with replacements indicated above, the performance of this optimum receiver, namely,

$$P_{e,o} = \operatorname{erfc}\left[\sqrt{E'(1-\rho')/4}\,\right] \tag{4.5}$$

(the subscript o denotes that receiver is optimum for the model used). Since

$$\frac{E(1-\rho)}{N_0} = \frac{1}{2N_0}\int_T s_d^2(t)\,dt$$

and from Equation 4.3 and Equation 4.4

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\int_T \frac{s_d^2(t)}{1 + q^2(t)\dfrac{N_0}{2}}\,dt \tag{4.6}$$

it is clear that

$$E'(1-\rho')/2 \le E(1-\rho)/N_0$$

so that $P_{e,o} \ge P_e$. This means that the optimum receiver designed to operate in a nonstationary interference environmemt, performs worse that the optimum receiver designed to operate in the presence of WGN only, even though the former stationarized the interference prior to performing the (standard) correlation operation. This is not surprising because the former receiver is operating at a higher level of interference than the latter receiver. (Observe that with $q(t) = 0$, $P_e = P_{e,o}$ as expected). Furthermore, we must have $P_{e,o} \le P_{e,s}$. This can be demonstrated by use of the Cauchy-Schwarz inequality. Assume this last inequality to be true. This would imply from Equation 3.7 and Equation 4.6 that

$$\left[\frac{1}{2}\int_T s_d^2(t)\,dt\right]^2 / \left\{\int_T \left[\frac{N_0}{2} + q^2(t)\right] s_d^2(t)\,dt\right\} \le \frac{1}{4}\int_T \frac{s_d^2(t)}{\dfrac{N_0}{2} + q^2(t)}\,dt$$

and rearranging this expression, we have

$$\left[\int_T s_d^2(t)\,dt \right]^2 \le \int_T \left[\frac{N_0}{2} + q^2(t) \right] s_d^2(t)\,dt \int_T \frac{s_d^2(t)}{\frac{N_0}{2} + q^2(t)}\,dt$$

which is precisely a form of the Cauchy-Schwarz inequality applied to the inner product of two functions $g_1(t)$ and $g_2(t)$, where in this case

$$g_1(t) = \left[\frac{N_0}{2} + q^2(t) \right]^{1/2} s_d(t)$$

$$g_2(t) = \frac{s_d(t)}{\left[\frac{N_0}{2} + q^2(t) \right]^{1/2}}.$$

The unsophisticated jammer case discussed in the previous chapter, where $q(t) = K$, in this case yields from Equations (4.3) - (4.5)

$$E' = \frac{E}{\frac{N_0}{2} + K^2} \quad ; \qquad \rho' = \frac{\frac{N_0}{2} + K^2}{E} \int_T \frac{s_0(t)s_1(t)}{\frac{N_0}{2} + K^2}\,dt = \rho$$

so that

$$P_{e,o} = \operatorname{erfc}\left[\sqrt{\frac{E(1-\rho)}{2(N_0 + 2K^2)}} \right] \qquad (4.7)$$

which is identical to Equation 3.10. This is again not surprising since the assumed conditions result in simple AWGN interference for which the receiver of Figure 1.1 is indeed optimum, hence the result that $P_{e,o} = P_{e,s}$.

V. ANALYSIS OF RECEIVER PERFORMANCES WITH EXAMPLE INTERFERENCE MODELS

The results of the two previous chapters are now used to evaluate receiver performance under various interference conditions for two important signaling schemes, namely Phase Reversal Keying (PRK) and Frequency Shift Keying (FSK). For PRK, the transmitted signals are

$$s_i(t) = Am \sin 2\pi f_r t - (-1)^i A \sqrt{1 - m^2} \cos 2\pi f_r t \qquad 0 \le t \le T,\ i=0,1 \qquad (5.1)$$

where $0 \le m < 1$. The vector diagram of Figure 5.1 shows that the parameter m controls the phase angle between the signals $s_0(t)$ and $s_1(t)$. When $m = 1$, the two signals are undistinguishable, whereas for $m = 0$, the angle between the signals is 180° and we have the familiar Phase Shift Keying (PSK) scheme. For FSK the transmitted signals are

$$s_i(t) = A \sin 2\pi f_i t \qquad 0 \le t \le T,\ i = 0,1. \qquad (5.2)$$

We assume for simplicity that $2f_r T$ as well as $2f_i T$ (for $i=0,1$) are integers. This means that

$$\int_0^T s_i^2(t)\, dt = A^2 T/2 \equiv E = \text{Energy per bit} \qquad i = 0,1 \qquad (5.3)$$

for both signaling schemes. Observe therefore that from Equation 1.3, $\rho = 2m^2 - 1$ for PRK and $\rho = 0$ for PSK.

Several interference conditions are now analyzed for the PRK and FSK signaling schemes introduced. For ease of comparison, all interferences are normalized so that they all have equal power (see Equation 2.4). Furthermore, we assume (as a worst case situation) that the interferer has timing information about the receiver (that is, the interference is synchronized to the receiver clock), and that it uses a waveform $q(t)$ (see Equation 2.1) that repeats itself every T seconds.

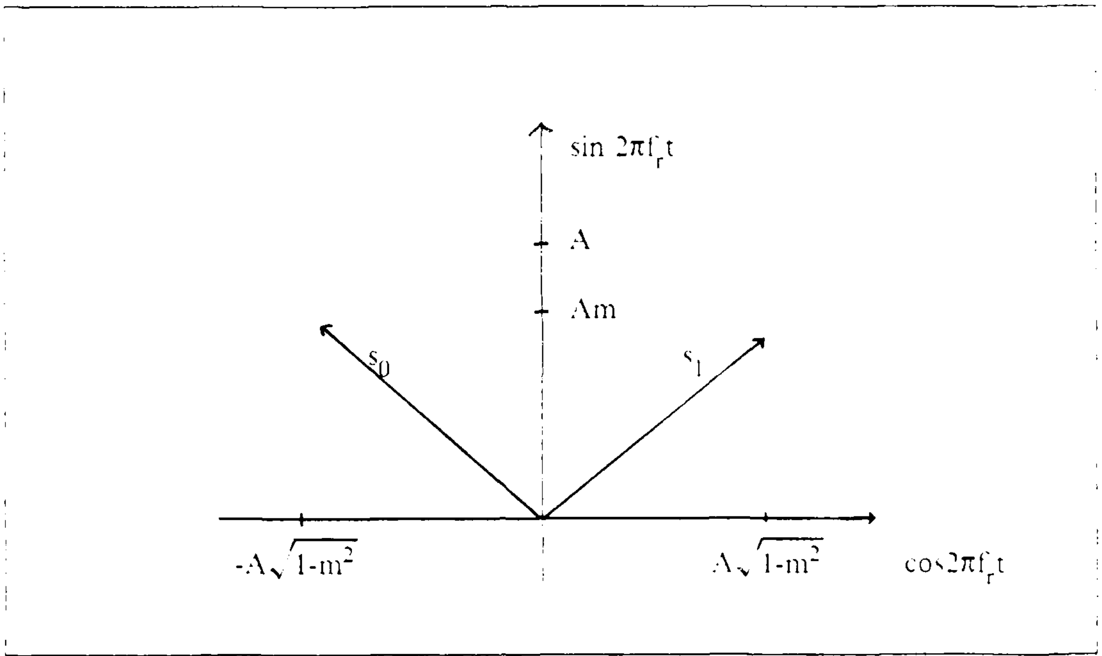


Figure 5.1 Vector Diagram of Signals $s_0(t)$ and $s_1(t)$.

A. LINEAR WAVE

We define

$$q(t) = \begin{cases} \dfrac{a}{T}t + b & 0 \le t \le cT \\ 0 & cT < t \le T \end{cases} \qquad 0 \le c \le 1 \qquad (5.4)$$

resulting in an interference generating waveform as depicted in Figure 5.2. From Equation 2.3 and Equation 2.4, we can obtain

$$Q_a(0) = \frac{1}{T}\int_0^{cT} [\, q(t)\,]^2\, dt = \frac{c}{3}\,(a^2c^2 + 3abc + 3b^2)$$

for the average power of the nonstationary interference.

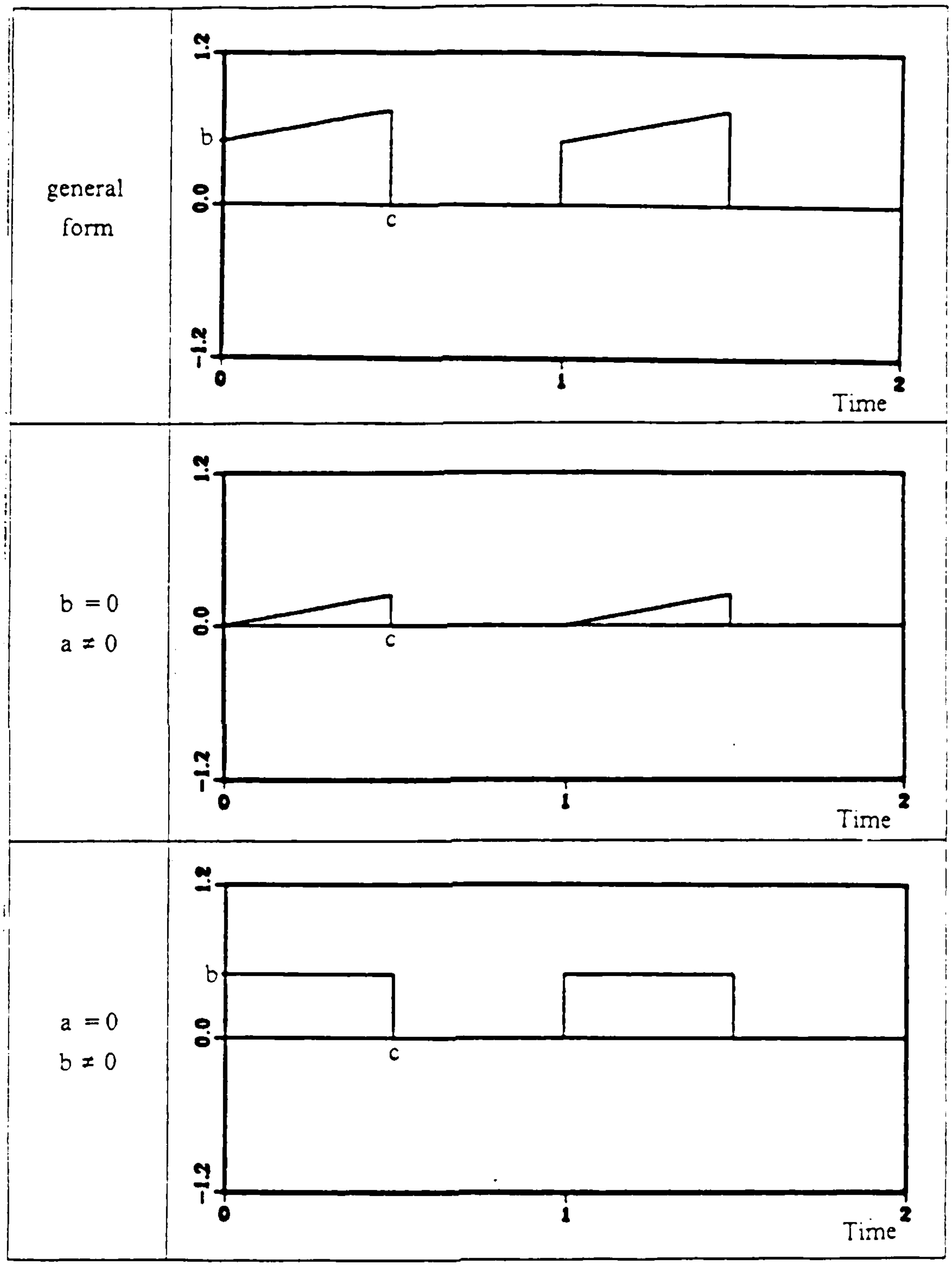


Figure 5.2 Several Variations of Linear Wave Interference as a Function of (normalized) Time.

1. **Phase Reversal Keying**

The linear wave interference of Equation 5.4 is now used to determine the receiver probability of error for PRK focusing first on the performance of the receiver of Figure 1.1. From Equation 5.1, we obtain

$$s_d(t) = 2A\sqrt{1 - m^2}\cos 2\pi f_r t \qquad 0 \le t \le T.$$

The following integral must first be evaluated in order to obtain $P_{e,s}$, namely

$$\int_T Q(t)s_d^2(t)dt = \int_0^{cT} \left(\frac{a^2t^2}{T^2} + \frac{2abt}{T} + b^2\right)[\,4A^2(1 - m^2)\cos^2 2\pi f_r t\,]\,dt$$

$$= 2A^2T(1 - m^2)\frac{c}{3}(a^2c^2 + 3abc + 3b^2)\cdot$$

$$\left[\frac{3(a^2c^2 + abc)\cos 4\pi f_r cT - 3abc + 2(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^2}{2(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^2}\right.$$

$$\left. - \frac{3a^2c^2\sin 4\pi f_r cT}{4(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^3} + \frac{3(a^2c^2 + 2abc + b^2)\sin 4\pi f_r cT}{2(a^2c^2 + 3abc + 3b^2)\,2\pi f_r cT}\right]. \tag{5.5}$$

a. Performance of the Suboptimum Receiver

From Equation 3.8 and Equation 5.5, we obtain

$$C = \frac{JSR\cdot SNR\cdot \varepsilon_1}{1 + JSR\cdot SNR\cdot \varepsilon_1} \tag{5.6}$$

where

$$JSR \triangleq \frac{Q_a(0)}{E} = \frac{c}{3}(a^2c^2 + 3abc + 3b^2)\,E$$

$$SNR \triangleq \frac{E}{N_0/2}$$

and

$$\varepsilon_1 = \left[\frac{3(a^2c^2 + abc)\cos 4\pi f_r cT - 3abc + 2(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^2}{2(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^2} - \frac{3a^2c^2 \sin 4\pi f_r cT}{4(a^2c^2 + 3abc + 3b^2)(2\pi f_r cT)^3} + \frac{3(a^2c^2 + 2abc + b^2)\sin 4\pi f_r cT}{2(a^2c^2 + 3abc + 3b^2)\, 2\pi f_r cT} \right].$$

This expression demonstrates that the performance of the conventional suboptimum receiver depends on the SNR as well as JSR and on the actual values of the parameters a, b, c, f_r, and T. The receiver designer has limited control of JSR. Clearly, as JSR $\rightarrow \infty$, C $\rightarrow$ 1 for constant SNR, and $P_{e,s} \rightarrow 1/2$ (see Equation 3.7). If the jammer is not present C = 0 and $P_{e,s} = P_e$. However, for constant JSR·SNR, the receiver designer can attempt to minimize C by proper choice of the product $f_r cT$. For fixed JSR, it is apparent that $\varepsilon_1 \rightarrow 0$ as $f_r T \rightarrow \infty$ (for c constant). Thus C $\rightarrow$ 0 as $f_r T \rightarrow \infty$ resulting in improved (suboptimum) receiver performance, namely $P_{e,s} \rightarrow P_e$. In practice, this means using as high a signaling frequency as possible, or choosing as long an observation time T as possible.

Actual performance (in terms of probability of error) for the conventional receiver is obtained from Equation 3.7. Observe that for PRK,

$$\frac{E(1-\rho)}{N_0} = \frac{A^2T(1-m^2)}{N_0} = \frac{E}{N_0/2}(1-m^2) = SNR(1-m^2) \qquad (5.7)$$

and from Equation 5.6

$$1 - C = \frac{1}{1 + JSR \cdot SNR \cdot \varepsilon_1}$$

so that

$$P_{e,s} = \operatorname{erfc}\left[\sqrt{SNR(1-m^2)\alpha/2}\right] \qquad (5.8)$$

where

$$\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_1).$$

A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.3 with m set to zero (resulting in PSK modulation), and parameters b set to zero and c set to one.

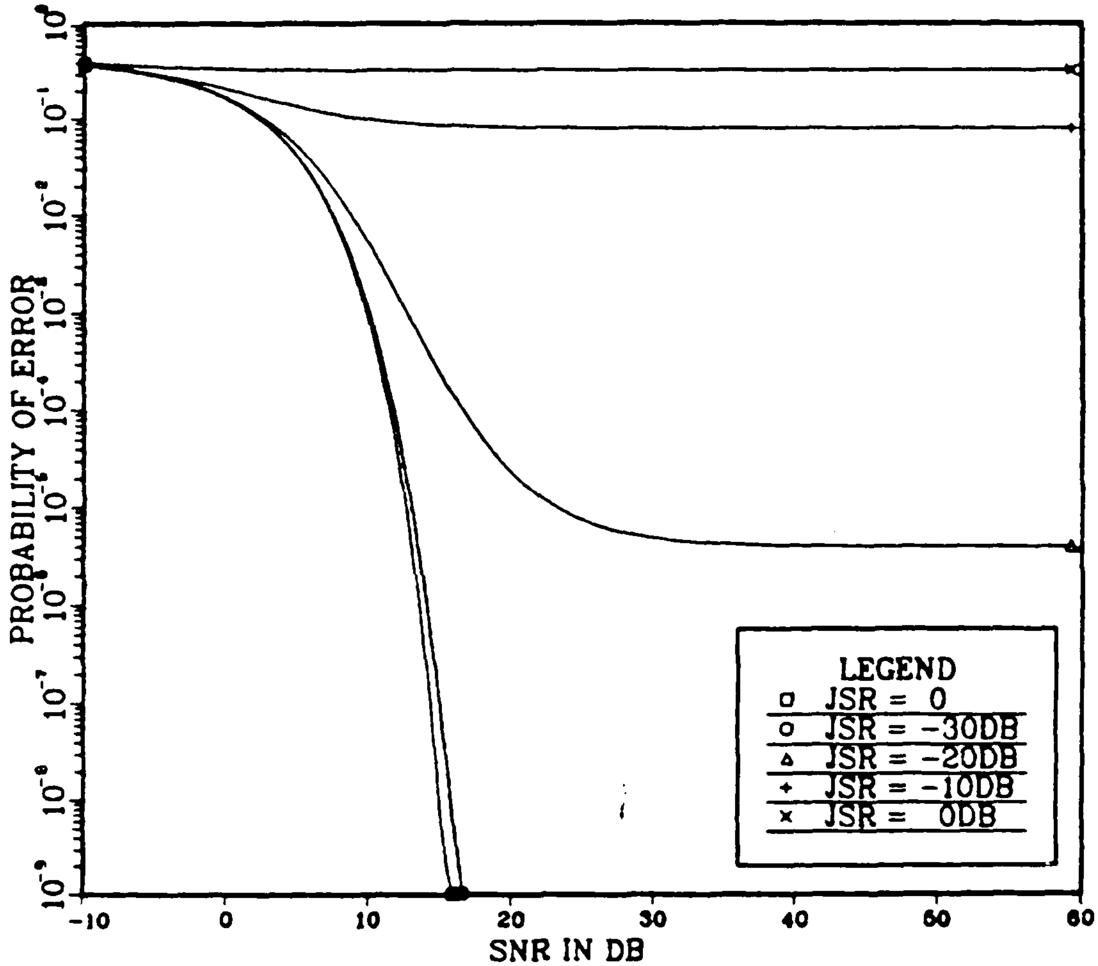


Figure 5.3 Performance of the Suboptimum Receiver for PSK Modulation with Linear Wave Interference.

b. Performance of the Optimum Receiver

The previous result for $P_{e,s}$ can be compared to the performance of the optimum receiver. From Equation 4.5 and Equation 4.6, we have

$$\frac{E'(1-\rho')}{2} = \frac{A^2T(1-m^2)}{N_0}\left[\int_0^c \frac{1+\cos 4\pi f_r T x}{1+(a^2x^2+2abx+b^2)\frac{N_0}{2}}\,dx \right.$$

$$\left. + \int_c^1 (1+\cos 4\pi f_r T x)\,dx \right] \tag{5.9}$$

where the change of variables $x = t/T$ has been used in Equation 5.9. If we let $b = 0$ and $c = 1$ as a special case, then $q(t) = at/T$, $0 \leq t \leq T$, $Q_a(0) = a^2/3$, and $JSR = (a^2/3)/E$ so that

$$\frac{E'(1-\rho')}{2} = \frac{A^2T(1-m^2)}{N_0}\int_0^1 \frac{1+\cos 4\pi f_r T x}{1+3JSR\cdot SNR\, x^2}\,dx.$$

Let $\sqrt{3JSR\cdot SNR}\, x = y$, then $dx = dy/\sqrt{3JSR\cdot SNR}$, so that

$$\frac{E'(1-\rho')}{2} = \frac{A^2T(1-m^2)}{N_0}\cdot\frac{1}{\sqrt{3JSR\cdot SNR}}\int_0^{\sqrt{3JSR\cdot SNR}} \frac{1+\cos(4\pi f_r T y/\sqrt{3JSR\cdot SNR})}{1+y^2}\,dy$$

which can be expressed in the form

$$\frac{E'(1-\rho')}{2} = \frac{SNR(1-m^2)}{\sqrt{3JSR\cdot SNR}}\left[\tan^{-1}\sqrt{3JSR\cdot SNR} + \int_0^{\sqrt{3JSR\cdot SNR}} \frac{\cos(4\pi f_r T y/\sqrt{3JSR\cdot SNR})}{1+y^2}\,dy\right] \tag{5.10}$$

where the remaining integral in Equation 5.10 can only be evaluated numerically or an approximation can be obtained, if it is assumed that $\sqrt{3JSR\cdot SNR} \gg 1$. (The derivation of such an approximation is presented in Appendix A). In the case of $JSR = 0$, then $q(t) = 0$, so that directly from Equation 4.6 we have

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\int_T s_d^2(t)\,dt = SNR(1-m^2) = \frac{E(1-\rho)}{N_0}. \qquad (5.11)$$

Thus we obtain

$$P_{e,o} = \begin{cases} \operatorname{erfc}\left[\sqrt{SNR(1-m^2)/2}\,\right] & JSR = 0 \\ \operatorname{erfc}\left[\sqrt{SNR(1-m^2)\beta/2}\,\right] & JSR \neq 0 \end{cases} \qquad (5.12)$$

where

$$\beta = \frac{1}{\sqrt{3JSR\cdot SNR}}\left[\tan^{-1}\sqrt{3JSR\cdot SNR} + \int_0^{\sqrt{3JSR\cdot SNR}} \frac{\cos\left(4\pi f_r T y\sqrt{3JSR\cdot SNR}\right)}{1+y^2}\,dy\right]$$

$$\approx \pi/\sqrt{3JSR\cdot SNR}\,.$$

A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.4 with m set to zero, and parameters b set to zero and c set to one.

Comparison of Equation 5.8 and Equation 5.12 is difficult at best because the parameter β can only be approximated (see Appendix A). We observe that with b = 0 and c = 1,

$$\alpha = \frac{1}{1 + JSR\cdot SNR\cdot\dfrac{3 + 2(2\pi f_r T)^2}{2(2\pi f_r T)^2}}.$$

If JSR is very large (with SNR fixed) and $f_r T$ is allowed to become unbounded (in order to the suboptimum receiver to overcome some of the jamming present), then

$$\alpha = 1/(1 + JSR\cdot SNR)$$

while

$$\beta \approx \frac{\pi}{\sqrt{3JSR \cdot SNR}}$$

is a close approximation, and clearly $\beta > \alpha$.

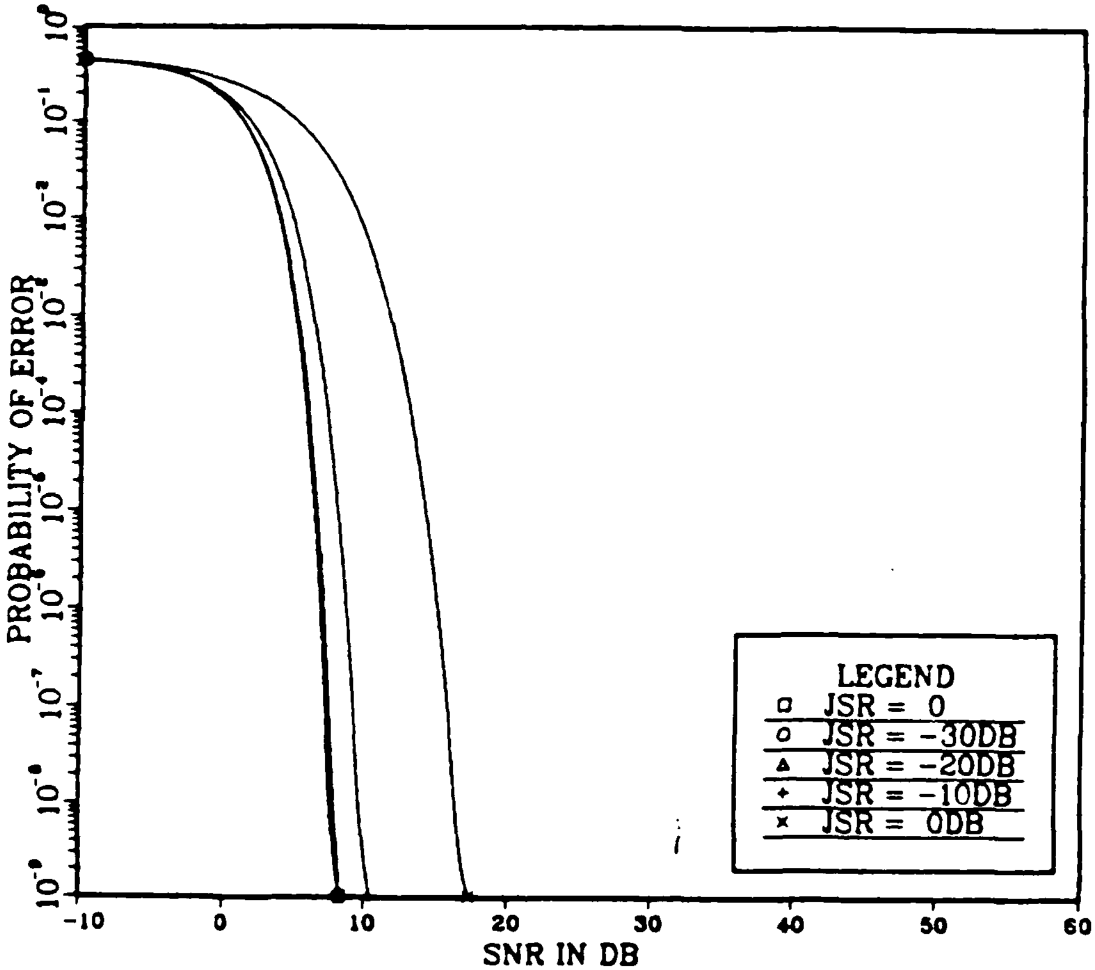


Figure 5.4 Performance of the Optimum Receiver for PSK Modulation with Linear Wave Interference.

For example, when JSR·SNR takes on a value of 15dB, $\beta/\alpha \approx 10$, demonstrating the fact that $P_{e,o} < P_{e,s}$ (as expected). One must however note that as JSR increase without bound, for fixed SNR, both α and β tend to zero resulting in both receivers displaying an increasingly worse performance, namely $P_{e,o} = P_{e,s} \rightarrow 0.5$.

2. **Frequency Shift Keying**

From Equation 5.2, we have

$$s_d(t) = 2A \cos \pi(f_1 + f_0)t \sin \pi(f_1 - f_0)t \qquad 0 \le t \le T$$

and we evaluate

$$\int_T Q(t) s_d^2(t)dt = \int_0^{cT} q^2(t) [\, 2A \cos \pi(f_1 + f_0)t \;\; \sin \pi(f_1 - f_0)t \,]^2 \, dt$$

$$= \frac{A^2Tc}{3}(a^2c^2 + 3abc + 3b^2)\cdot$$

$$\{1 - \frac{3(a^2c + ab)}{\pi^2 c(a^2c^2 + 3abc + 3b^2)}[\frac{2\cos \pi(\alpha_1 - \alpha_0)c}{(\alpha_1 - \alpha_0)^2} + \frac{\cos 2\pi\alpha_1 c}{(2\alpha_1)^2} + \frac{\cos 2\pi\alpha_0 c}{(2\alpha_0)^2} - \frac{2\cos \pi(\alpha_1 + \alpha_0)c}{(\alpha_1 + \alpha_0)^2}]$$

$$- \frac{3(a^2c^2 + 2abc + b^2)}{2\pi c(a^2c^2 + 3abc + 3b^2)}[\frac{2\sin \pi(\alpha_1 - \alpha_0)c}{\alpha_1 - \alpha_0} + \frac{\sin 2\pi\alpha_1 c}{2\alpha_1} + \frac{\sin 2\pi\alpha_0 c}{2\alpha_0} - \frac{2\sin \pi(\alpha_1 + \alpha_0)c}{\alpha_1 + \alpha_0}]$$

$$+ \frac{3a^2}{\pi^3 c(a^2c^2 + 3abc + 3b^2)}[\frac{2\sin \pi(\alpha_1 - \alpha_0)c}{(\alpha_1 - \alpha_0)^3} + \frac{\sin 2\pi\alpha_1 c}{(2\alpha_1)^3} + \frac{\sin 2\pi\alpha_0 c}{(2\alpha_0)^3} - \frac{2\sin \pi(\alpha_1 + \alpha_0)c}{(\alpha_1 + \alpha_0)^3}]$$

$$+ \frac{3ab}{\pi^2 c(a^2c^2 + 3abc + 3b^2)}[\frac{2}{(\alpha_1 - \alpha_0)^2} + \frac{1}{(2\alpha_1)^2} + \frac{1}{(2\alpha_0)^2} - \frac{2}{(\alpha_1 + \alpha_0)^2}]\} \qquad (5.13)$$

where $\alpha_i = 2f_iT$, $i = 0,1$.

a. Performance of the Suboptimum Receiver

From Equation 3.8 and Equation 5.13 we obtain

$$C = \frac{JSR \cdot SNR \cdot \varepsilon_1}{1 + JSR \cdot SNR \cdot \varepsilon_1} \qquad (5.14)$$

where as before

$$JSR = [\, \frac{c}{3}(a^2c^2 + 3abc + 3b^2) \,] \, I.$$

$$SNR = \frac{E}{N_0 2}$$

and

$$\varepsilon_1' = \{1$$
$$- \frac{3(a^2c+ab)}{\pi^2 c(a^2c^2+3abc+3b^2)}\left[\frac{2\cos\pi(\alpha_1-\alpha_0)c}{(\alpha_1-\alpha_0)^2}+\frac{\cos 2\pi\alpha_1 c}{(2\alpha_1)^2}+\frac{\cos 2\pi\alpha_0 c}{(2\alpha_0)^2}-\frac{2\cos\pi(\alpha_1+\alpha_0)c}{(\alpha_1+\alpha_0)^2}\right]$$
$$- \frac{3(a^2c^2+2abc+b^2)}{2\pi c(a^2c^2+3abc+3b^2)}\left[\frac{2\sin\pi(\alpha_1-\alpha_0)c}{\alpha_1-\alpha_0}+\frac{\sin 2\pi\alpha_1 c}{2\alpha_1}+\frac{\sin 2\pi\alpha_0 c}{2\alpha_0}-\frac{2\sin\pi(\alpha_1+\alpha_0)c}{\alpha_1+\alpha_0}\right]$$
$$+ \frac{3a^2}{\pi^3 c(a^2c^2+3abc+3b^2)}\left[\frac{2\sin\pi(\alpha_1-\alpha_0)c}{(\alpha_1-\alpha_0)^3}+\frac{\sin 2\pi\alpha_1 c}{(2\alpha_1)^3}+\frac{\sin 2\pi\alpha_0 c}{(2\alpha_0)^3}-\frac{2\sin\pi(\alpha_1+\alpha_0)c}{(\alpha_1+\alpha_0)^3}\right]$$
$$+ \frac{3ab}{\pi^2 c(a^2c^2+3abc+3b^2)}\left[\frac{2}{(\alpha_1-\alpha_0)^2}+\frac{1}{(2\alpha_1)^2}+\frac{1}{(2\alpha_0)^2}-\frac{2}{(\alpha_1+\alpha_0)^2}\right]\}.$$

In order to maximize C, ε_1' must be minimized, with the previously introduced restriction on α_1 and α_0 that they be integers. For b = 0, and c = 1,

$$\varepsilon_1' = 1 - 3\left[\frac{2\cos\pi(\alpha_1-\alpha_0)c}{\pi^2(\alpha_1-\alpha_0)^2}+\frac{\cos 2\pi\alpha_1 c}{(2\pi\alpha_1)^2}+\frac{\cos 2\pi\alpha_0 c}{(2\pi\alpha_0)^2}-\frac{2\cos\pi(\alpha_1+\alpha_0)c}{\pi^2(\alpha_1+\alpha_0)^2}\right] \quad (5.15)$$

so that minimization of ε_1' can be accomplished by maximizing the term in brackets in Equation 5.15. It appears that optimum choices for these parameters are values that result in $\alpha_1 - \alpha_0$ being small. Table 1 shows that when $\alpha_1 = 3$, $\alpha_0 = 1$, c = 1 and b = 0, then $\varepsilon_1' = 0.801579349$, the minimum value achievable. Thus the optimum value of C at fixed JSR·SNR is

$$C = \frac{0.8016\ JSR\cdot SNR}{1 - 0.8016\ JSR\cdot SNR} \quad (5.16)$$

which can be achieved with finite values of f_r and T, unlike the PRK scheme, where infinite frequency-time products are required for optimum results.

TABLE 1

COMPUTED VALUES OF ε_1' FOR GIVEN VALUES OF α_1 AND α_0

($\alpha_d = \alpha_1 - \alpha_0$)

α_1	ε_1' ($\alpha_d = 1$)
2	1.4454
3	1.5562
4	1.5823
5	1.5926
6	1.5978
7	1.6007
8	1.6025
9	1.6037
10	1.6045

α_1	ε_1' ($\alpha_d = 3$)
4	0.9625
5	1.0331
6	1.0495
7	1.0562
8	1.0597
9	1.0618
10	1.0631
11	1.0640
12	1.0647

α_1	ε_1' ($\alpha_d = 2$)
3	0.8016
4	0.8412
5	0.8460
6	0.8472
7	0.8476
8	0.8478
9	0.8479
10	0.8479
11	0.8480

α_1	ε_1' ($\alpha_d = 4$)
5	0.8999
6	0.9054
7	0.9581
8	0.9603
9	0.9611
10	0.9615
11	0.9617
12	0.9618
13	0.9619

α_d	ε_1' ($\alpha_0 = 1$)
5	0.9338
6	0.9151
7	0.9277
8	0.9197
9	0.9257

α_d	ε_1 ($\alpha_0 = 1$)
10	0.9215
11	0.9249
12	0.9224
13	0.9245

Actual performance (in terms of probability of error) for the conventional receiver is obtained from Equation 3.7. Observe that for FSK

$$\frac{E(1-\rho)}{N_0} = \frac{E/2}{N_0/2} = SNR/2 \qquad (5.17)$$

and

$$1 - C = \frac{1}{1 + JSR \cdot SNR \cdot \varepsilon_1'}$$

so that

$$P_{e,s} = \text{erfc}\left(\sqrt{SNR\,\alpha/4}\right) \qquad (5.18)$$

where

$$\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_1').$$

A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.5 with parameters b set to zero, and c set to one.

b. Performance of the Optimum Receiver

Again, the above result for the performance of the suboptimum receiver can be compared to the performance of the optimum receiver using Equation 4.5 and Equation 4.6. For the FSK modulation case being considered and linear wave interference,

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\left\{\int_0^{cT} \frac{[2A\cos\pi(f_1+f_0)t\,\sin\pi(f_1-f_0)t]^2}{1 + \left(\frac{a^2}{T^2}t^2 + 2abt + b^2\right)\frac{N_0}{2}}\,dt\right.$$

$$\left. + \int_{cT}^{T} [\,2A\cos\pi(f_1+f_0)t\,\sin\pi(f_1-f_0)t\,]^2\,dt\,\right\}.$$

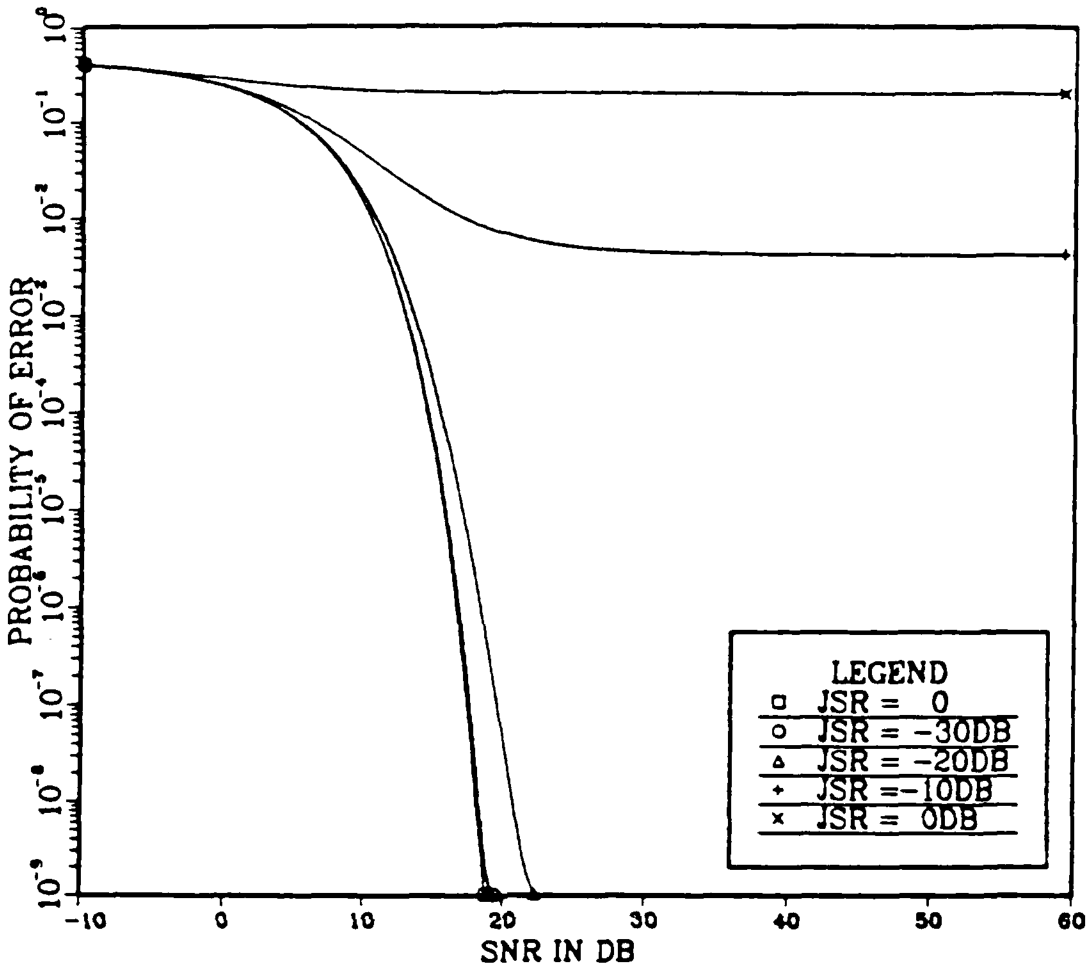


Figure 5.5 Performance of the Suboptimum Receiver for FSK Modulation with Linear Wave Interference.

This expression can be put in compact form with b set to zero and c set to one as a special case. Some algebraic manipulations yield

$$\frac{E(1-\rho)}{2} = \left(\frac{SNR}{2}\right)\cdot \left\{ \frac{4}{\sqrt{3JSR\cdot SNR}} \int_0^{\sqrt{3JSR\cdot SNR}} \frac{\left[\cos\left(\frac{\pi}{2}(\alpha_1-\alpha_0)x\sqrt{3JSR\cdot SNR}\right)\sin\left(\frac{\pi}{2}(\alpha_1-\alpha_0)x\sqrt{3JSR\cdot SNR}\right)\right]^2}{1+x^2}\, dx\right\}.$$

(5.19)

This equation can only be evaluated numerically for set values of the parameters α_1 and α_0, and changing values of JSR·SNR. When $\alpha_1 = 3$ and $\alpha_0 = 1$ (which is the optimum choice for mininizing C and thus minimizing $P_{e,s}$). Equation 5.19 becomes

$$\frac{E'(1-\rho')}{2} = \frac{2SNR}{\sqrt{3JSR \cdot SNR}} \int_0^{\sqrt{3JSR \cdot SNR}} \frac{[\cos(2\pi x\sqrt{3JSR \cdot SNR}) \ \sin(\pi x\sqrt{3JSR \cdot SNR})]^2}{1+x^2}\, dx]. \tag{5.20}$$

If JSR = 0, then q(t) = 0, therefore $E'(1-\rho')/2 = E(1-\rho)/N_0$. Thus

$$\frac{E'(1-\rho')}{2} = \frac{E/2}{N_0/2} = SNR/2 \tag{5.21}$$

so that

$$P_{e,o} = \begin{cases} \operatorname{erfc}(\sqrt{SNR/4}) & JSR = 0 \\ \operatorname{erfc}(\sqrt{SNR\,\beta/4}) & JSR \neq 0 \end{cases} \tag{5.22}$$

where

$$\beta = \frac{4}{\sqrt{3JSR \cdot SNR}} \int_0^{\sqrt{3JSR \cdot SNR}} \frac{[\cos(2\pi y\sqrt{3JSR \cdot SNR}) \ \sin(\pi y\sqrt{3JSR \cdot SNR})]^2}{1+y^2}\, dy.$$

A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.6 with parameters b set to zero, and c set to one. Results of numerical evaluation of β are presented in Table 2 for b = 0 and c = 1, and compared with the values α.

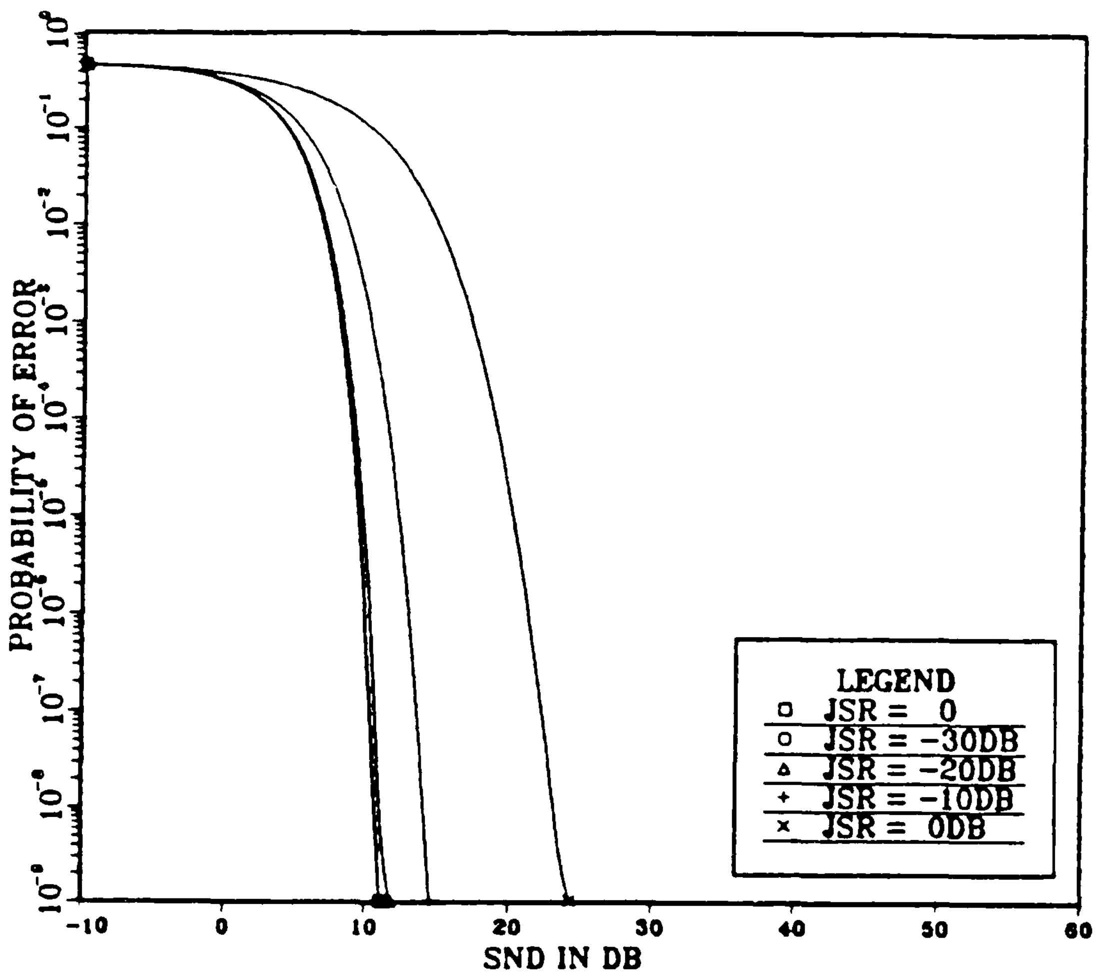


Figure 5.6 Performance of the Optimum Receiver for FSK Modulation with Linear Wave Interference.

3. **Comparison**

Performance comparison of the receivers for PRK and FSK with linear wave interference, from Equation 5.6 and Equation 5.16 shows that the parameter C for FSK does not grow as rapidly to 1 as a function of JSR·SNR, as does C for PRK. Thus, the frequency diversity of FSK provides a small jamming margin over PRK.

We note that differences between the parameters β and α are much smaller for FSK than differences between the corresponding factors β and α in the PRK scheme. Hence suboptimality is somewhat reduced in the FSK scheme.

TABLE 2

COMPARISON OF VALUES OF α AND β FOR GIVEN VALUES OF JSR·SNR

JSR·SNR	2	10	50	100	200	1,000
α	0.3841	0.1109	0.0243	0.0123	0.0062	0.0012
β	0.4214	0.1435	0.0387	0.0212	0.0114	0.0026
$\beta\ \alpha$	1.0970	1.2938	1.5898	1.7205	1.8390	2.0867

B. SINUSOIDAL WAVE

We define

$$q(t) = \begin{cases} \sqrt{\dfrac{2}{3}}\, ac \cos 2\pi f_c t + \sqrt{abc + b^2} & 0 \le t \le cT \\ 0 & cT < t \le T \end{cases} \qquad 0 \le c \le 1 \tag{5.23}$$

as another interference generating function which can be depicted as shown in Figure 5.7. Then from Equation 2.3 and Equation 5.23 we obtain

$$Q_a(0) = \frac{1}{T}\int_0^{cT} [\, q(t)\,]^2\, dt$$

$$= \frac{c}{3}\,[\, a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT + 2ac\sqrt{6(abc + b^2)} \text{ sinc } 2f_c cT\,]$$

for the average power of the nonstationary interference. Observe that whenever $2f_c cT$ is an integer, $Q_a(0)$ above is identical to the average power of the linear wave interference.

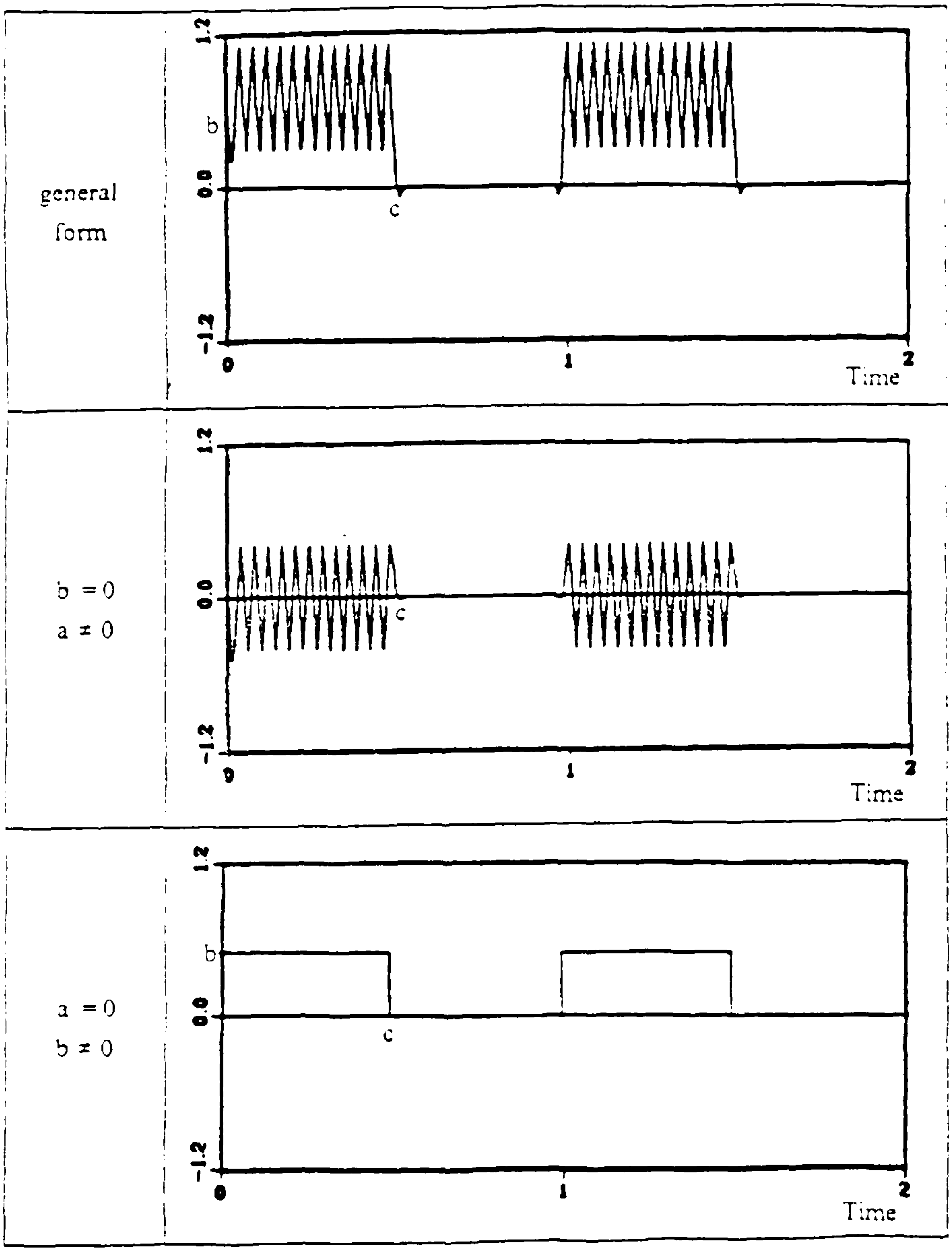


Figure 5.7 Several Variations of Sinusoidal Wave Interference as a Function of (normalized) Time.

1. **Phase Reversal Keying**

For PRK, from Equation 5.1 we obtain again

$$s_d(t) = 2A\sqrt{1-m^2}\cos 2\pi f_r t \qquad\qquad 0 \le t \le T.$$

Prior to obtaining receiver performance we evaluate the integral

$$\int Q(t)s_d^2(t)dt = \int_0^{cT} q^2(t)\,4A^2(1-m^2)\cos^2 2\pi f_r t\,dt$$

$$= 2A^2T(1-m^2)\frac{c}{3}[\,a^2c^2 + 3abc + 3b^2 + a^2c^2\text{sinc}\,4f_c cT + 2ac\sqrt{6(abc+b^2)}\,\text{sinc}\,2f_c cT\,]\cdot$$

$$\{1 + \frac{0.5a^2c^2[\,2\text{sinc}\,4f_r cT + \text{sinc}\,4(f_c-f_r)cT + \text{sinc}\,4(f_c+f_r)cT\,]}{a^2c^2 + 3abc + 3b^2 + a^2c^2\,\text{sinc}\,4f_c cT + 2ac\sqrt{6(abc+b^2)}\,\text{sinc}\,2f_c cT}$$

$$+ \frac{ac\sqrt{6(abc+b^2)}\,[\,\text{sinc}\,2(f_c-2f_r)cT + \text{sinc}\,2(f_c+2f_r)cT\,]}{a^2c^2 + 3abc + 3b^2 + a^2c^2\,\text{sinc}\,4f_c cT + 2ac\sqrt{6(abc+b^2)}\,\text{sinc}\,2f_c cT}$$

$$+ \frac{3(abc+b^2)\,\text{sinc}\,4f_r cT}{a^2c^2 + 3abc + 3b^2 + a^2c^2\,\text{sinc}\,4f_c cT + 2ac\sqrt{6(abc+b^2)}\,\text{sinc}\,2f_c cT}\}. \tag{5.24}$$

where

$$\text{sinc}\,(x) = \frac{\sin \pi x}{\pi x}.$$

a. Performance of the Suboptimum Receiver

From Equation 3.8 and Equation 5.23, we obtain

$$C = \frac{JSR\cdot SNR\cdot \varepsilon_s}{1 + JSR\cdot SNR\cdot \varepsilon_s} \tag{5.25}$$

where

$$\mathrm{JSR} = \frac{\frac{c}{3}\left[\, a^2c^2 + 3abc + 3b^2 + a^2c^2 \operatorname{sinc} 4f_c cT + 2ac\sqrt{6(abc+b^2)} \operatorname{sinc} 2f_c cT \,\right]}{E}$$

$$\mathrm{SNR} = \frac{E}{N_0\, 2}$$

and

$$\begin{aligned}\varepsilon_s = \{1 &+ \frac{0.5a^2c^2[\, 2\operatorname{sinc} 4f_r cT + \operatorname{sinc} 4(f_c - f_r)cT + \operatorname{sinc} 4(f_c + f_r)cT \,]}{a^2c^2 + 3abc + 3b^2 + a^2c^2 \operatorname{sinc} 4f_c cT + 2ac\sqrt{6(abc+b^2)} \operatorname{sinc} 2f_c cT} \\ &+ \frac{ac\sqrt{6(abc+b^2)}\,[\, \operatorname{sinc} 2(f_c - f_r)cT + \operatorname{sinc} 2(f_c + f_r)cT \,]}{a^2c^2 + 3abc + 3b^2 + a^2c^2 \operatorname{sinc} 4f_c cT + 2ac\sqrt{6(abc+b^2)} \operatorname{sinc} 2f_c cT} \\ &+ \frac{3(abc+b^2) \operatorname{sinc} 4f_r cT}{a^2c^2 + 3abc + 3b^2 + a^2c^2 \operatorname{sinc} 4f_c cT + 2ac\sqrt{6(abc+b^2)} \operatorname{sinc} 2f_c cT}\}.\end{aligned} \tag{5.26}$$

For the special case in which $b = 0$ and $c = 1$, then $\varepsilon_s = 1$ for $f_c \neq f_r$ and $\varepsilon_s = 1.5$ for $f_c = f_r$, thus we have

$$C = \begin{cases} \mathrm{JSR{\cdot}SNR}\,(1 + \mathrm{JSR{\cdot}SNR}) & \text{if } f_c \neq f_r, \\ 3\mathrm{JSR{\cdot}SNR}\,(2 + 3\mathrm{JSR{\cdot}SNR}) & \text{if } f_c = f_r. \end{cases}$$

It is apparent that C is largest when $f_c = f_r$ for fixed JSR·SNR. This means (as could be expected) that the jammer is most damaging to the receiver when it operates at the signal frequency, and furthermore there is nothing the suboptimum receiver can do (in terms of choosing longer integration times for instance) in order to reduce jamming effects, other than to switch to a new frequency of operation.

Actual receiver performance is obtained from Equation 3.7. Observe that for PRK

$$\frac{E(1-\rho)}{N_0} = \frac{A^2T}{N_0}(1-m^2) = SNR(1-m^2) \tag{5.27}$$

and

$$1 - C = 1/(1 + JSR \cdot SNR \cdot \varepsilon_s)$$

so that

$$P_{e,s} = \text{erfc}\,[\,\sqrt{SNR(1-m^2)\alpha/2}\,] \tag{5.28}$$

where

$$\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_s).$$

A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.8 with m set to zero, $f_c = f_r$ and the parameters b set to zero, and c set to one.

b. Performance of the Optimum Receiver

Again, we compare previous result with the performance of the optimum receiver using Equation 4.5 and Equation 4.6. We obtain

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\int_T \frac{4A^2(1-m^2)\cos^2 2\pi f_r t}{1 + q^2(t)\dfrac{N_0}{2}}\,dt$$

$$= \frac{1}{2N_0}\Big\{\int_0^{cT} \frac{4A^2(1-m^2)\cos^2 2\pi f_r t}{1 + [\,\frac{a^2c^2}{3}(1+\cos 4\pi f_c t) + abc + b^2 + \sqrt{\frac{a^2c^2}{3}}(abc+b^2)\cos 2\pi f_c t\,]\,\frac{N_0}{2}}\,dt$$

$$+ \int_{cT}^{T} [\,4A^2(1-m^2)\cos^2 2\pi f_r t\,]\,dt\Big\}.$$

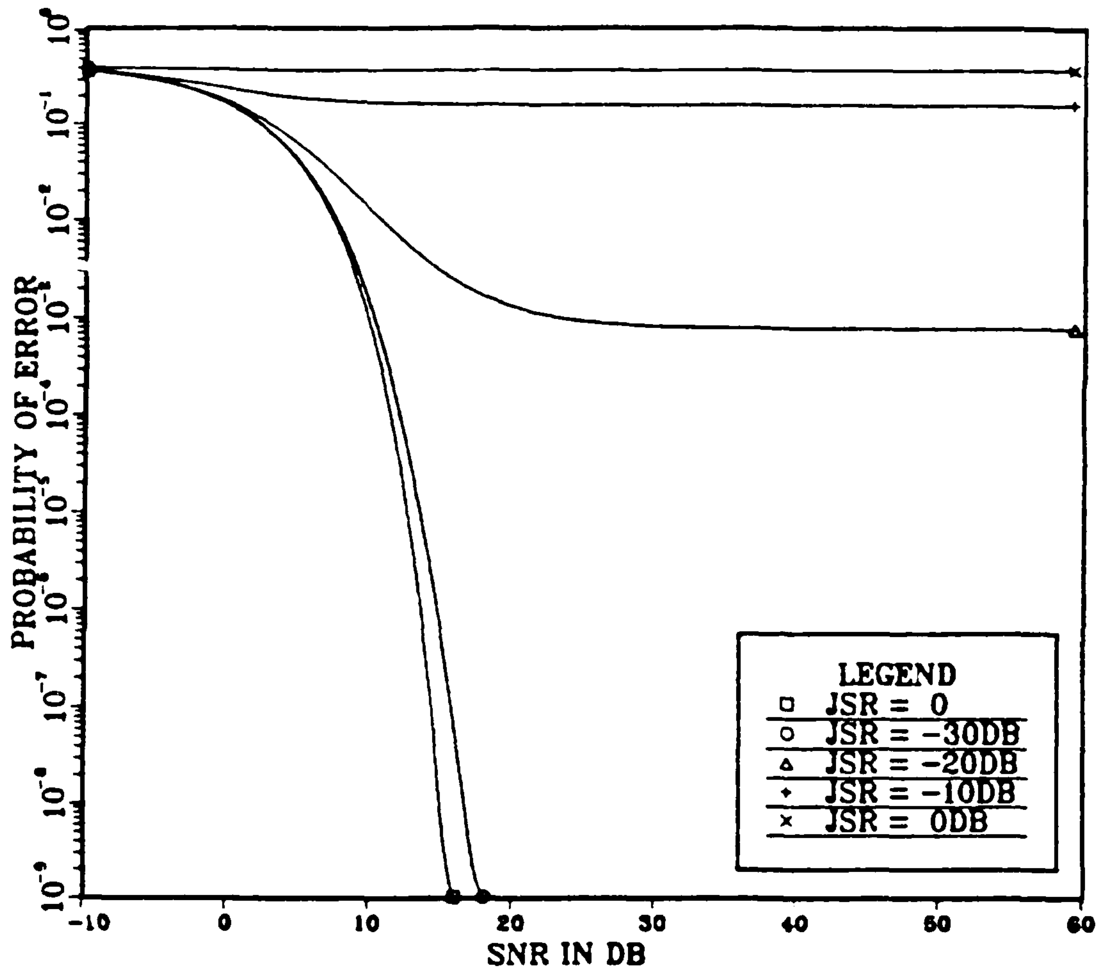


Figure 5.8 Performance of the Suboptimum Receiver for PSK Modulation with Sinusoidal Wave Interference.

If we assume that $b = 0$ and $c = 1$ as a special case, then $q(t) = \sqrt{2a^2/3}\cos 2\pi f_c t$, $Q_a(0) = a^2/3$, and $JSR = (a^2/3)/E$ so that

$$\frac{E(1-\rho)}{2} = \frac{1}{2N_0}\int_0^T \frac{4A^2(1-m^2)\cos^2 2\pi f_r t}{1 + \frac{a^2}{3}(1-\cos 4\pi f_c t)\frac{N_0}{2}}\,dt$$

$$= \frac{A^2(1-m^2)\eta}{2(a^2/3)}\int_0^T \frac{1 + \cos 4\pi f_r t}{1 + \eta\cos 4\pi f_c t}\,dt \qquad (5.29)$$

where

$$\eta = \frac{a^2/3}{a^2/3 + N_0/2} = \frac{JSR \cdot SNR}{1 + JSR \cdot SNR} \qquad ; \quad 0 \le \eta \le 1.$$

In order to evaluate the integral in closed form, we use the tabulated integral [Ref. 3: pp. 107-122],

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos q\, dx}{1 + \eta \cos x} = (-p)^q \frac{1 + p^2}{1 - p^2} \;,\; p = \frac{\eta}{1 + \sqrt{1 - \eta^2}} \qquad 0 \le \eta \le 1. \quad (5.30)$$

The integral in Equation 5.29 can be put in the form

$$\int_0^T \frac{1 + \cos 4\pi f_r t}{1 + \eta \cos 4\pi f_c t}\, dt = \int_0^{4\pi f_c T} \frac{1 + \cos qx}{1 + \eta \cos x} \cdot \frac{dx}{4\pi f_c} \qquad ; q = \frac{f_r}{f_c}.$$

In order to obtain a closed form expression for this integral, we assume that q is an integer (The most interesting case is $q = 1$, so the restriction is not too unreasonable). With this assumption, the integrand is periodic of period 2π, and since $4\pi f_c T = 2\pi \cdot 2f_c T$ and $2f_c T$ is assumed to be an integer, say $2f_c T = n$, then

$$\int_0^T \frac{1 + \cos 4\pi f_r t}{1 + \eta \cos 4\pi f_c t}\, dt = \frac{1}{4\pi f_c}\int_0^{2\pi n} \frac{1 + \cos qx}{1 + \eta \cos x}\, dx$$

$$= \frac{T}{2f_c T} \cdot \frac{n}{2\pi} \int_0^{2\pi} \frac{1 + \cos qx}{1 + \eta \cos x}\, dx = T\left[\, 1 + (-p)^q \,\right] \frac{1 + p^2}{1 - p^2}$$

and finally

$$\frac{E(1 - \rho^2)}{2} = SNR(1 - m^2)(1 - \eta)\left[\, 1 + (-p)^q \,\right] \frac{1 + p^2}{1 - p^2}$$

$$= SNR(1 - m^2)\left[\, 1 + \left(\frac{-\eta}{1 + \sqrt{1 - \eta^2}}\right)^q \frac{(1 - \eta)(1 + \sqrt{1 - \eta^2})}{1 - \eta^2 + \sqrt{1 - \eta^2}} \,\right]$$

$$= SNR(1 - m^2)\, \nu \qquad (5.31)$$

where

$$\mathbf{v} = \frac{1 + \sqrt{1 + 2JSR \cdot SNR}}{(1 + 2JSR \cdot SNR) + (1 + JSR \cdot SNR)\sqrt{1 + 2JSR \cdot SNR}}$$

If JSR = 0 then q(t) = 0, so that directly from Equation 4.6 we have

$$\frac{E(1-\rho)}{2} = \frac{A^2 T}{N_0}(1 - m^2). \tag{5.32}$$

Thus we obtain probability of error for the optimum receiver, namely

$$P_{e,o} = \begin{cases} \operatorname{erfc}\left[\sqrt{SNR(1-m^2)/2}\,\right] & JSR = 0 \\ \operatorname{erfc}\left[\sqrt{SNR(1-m^2)\mathbf{v}/2}\,\right] & JSR \neq 0 \end{cases} \tag{5.33}$$

where $\mathbf{v}$ is defined following Equation 5.31 A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.9 with m set to zero, and the parameters b set to zero, with c set to one.

If we compare the parameters $\mathbf{v}$ and $\boldsymbol{\alpha}$ when $f_r = f_c$ (i.e., q = 1), where b = 0, and c = 1, we find that

$$\boldsymbol{\alpha} = \frac{2}{2 + 3JSR \cdot SNR} < \frac{1 + \sqrt{1 + 2JSR \cdot SNR}}{(1 + 2JSR \cdot SNR) + (1 + JSR \cdot SNR)\sqrt{1 + 2JSR \cdot SNR}} = \mathbf{v} \tag{5.34}$$

Observe that this inequality becomes stronger as JSR increases with constant SNR. For instance at a JSR·SNR of 15dB, the two sides differ by a factor of 1.34, and at a JSR·SNR of 25dB, the factor becomes 1.44. Nevertheless, both sides of the inequality tend to zero as JSR → ∞, implying both $P_{e,s}$ and $P_{e,o}$ → 0.5 under that conditions. The optimum receiver has a definite "advantage" over suboptimum receiver for high JSR, however this "advantage" has a limit as $\mathbf{v}/\boldsymbol{\alpha} \to 1.5$ as JSR → ∞ in Equation 5.34.

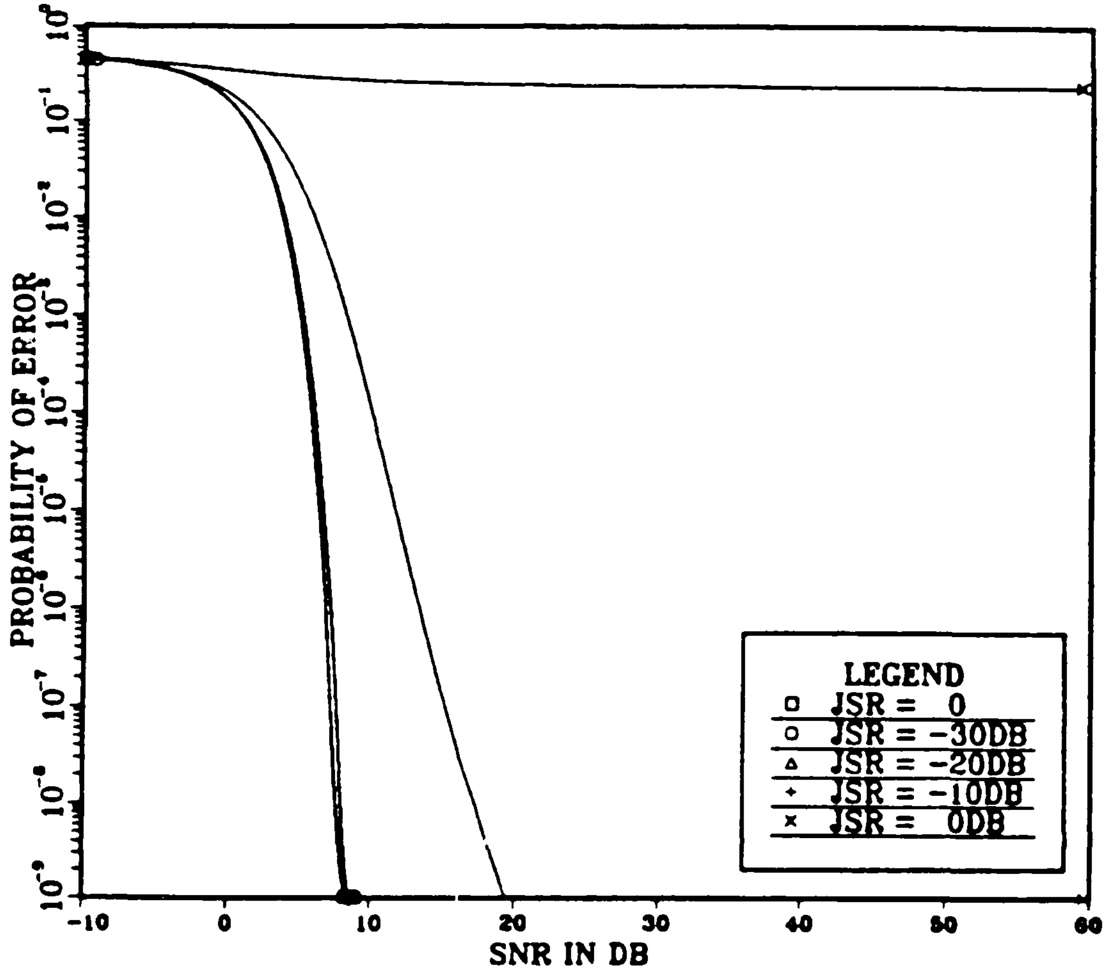


Figure 5.9 Performance of the Optimum Receiver for PSK Modulation with Sinusoidal Wave Interference.

2. **Frequency Shift Keying**

The above result for the performance of the suboptimum receiver can be compared to the performance of the optimum receiver using Equation 4.5 and Equation 4.6. For the FSK, from Equation 5.2 we have

$$s_J(t) = 2A \cos \pi(f_1 + f_0)t \sin \pi(f_1 - f_0)t \qquad 0 \le t \le T$$

and first evaluate

$$\int_T Q(t)s_d^2(t)\,dt = \int_0^{cT} q^2(t)[\,2A\cos\pi(f_1+f_0)t\sin\pi(f_1-f_0)t\,]^2\,dt$$

$$= A^2T\,\frac{c}{3}[\,a^2c^2+3abc+3b^2+a^2c^2\ \mathrm{sinc}\ 4f_ccT\ +2ac\sqrt{6(abc+b^2)}\ \mathrm{sinc}\ 2f_ccT\,]\,\cdot$$

$$[1+\frac{(a^2c^2+3abc+3b^2)(2\mathrm{sinc}\ 2f_scT-2\mathrm{sinc}\ 2f_dcT+\mathrm{sinc}\ 4f_1cT+\mathrm{sinc}\ 4f_0cT)}{2(a^2c^2+3abc+3b^2+a^2c^2\ \mathrm{sinc}\ 4f_ccT+2ac\sqrt{6(abc+b^2)}\ \mathrm{sinc}\ 2f_ccT)}$$

$$+a^2c^2[\frac{\mathrm{sinc}\ 2(2f_c-f_s)cT+\mathrm{sinc}\ 2(2f_c+f_s)cT-\mathrm{sinc}\ 2(2f_c-f_d)cT-\mathrm{sinc}\ 2(2f_c+f_d)cT}{2(a^2c^2+3abc+3b^2+a^2c^2\ \mathrm{sinc}\ 4f_ccT+2ac\sqrt{6(abc+b^2)}\ \mathrm{sinc}\ 2f_ccT)}$$

$$+\frac{\mathrm{sinc}\ 4(f_c-f_1)cT+\mathrm{sinc}\ 4(f_c+f_1)cT+\mathrm{sinc}\ 4(f_c-f_0)cT+\mathrm{sinc}\ 4(f_c+f_0)cT}{4(a^2c^2+3abc+3b^2+a^2c^2\ \mathrm{sinc}\ 4f_ccT+2ac\sqrt{6(abc+b^2)}\ \mathrm{sinc}\ 2f_ccT)}]$$

$$+2ac\sqrt{6(abc+b^2)}[\frac{\mathrm{sinc}\ 2(f_c-f_s)cT+\mathrm{sinc}\ 2(f_c+f_s)cT-\mathrm{sinc}\ 2(f_c-f_d)cT-\mathrm{sinc}\ 2(f_c+f_d)cT}{2(a^2c^2+3abc+3b^2+a^2c^2\mathrm{sinc}4f_ccT+2ac\sqrt{6(abc+b^2)}\mathrm{sinc}2f_ccT)}$$

$$+\frac{\mathrm{sinc}\ 2(f_c-2f_1)cT+\mathrm{sinc}\ 2(f_c+2f_1)cT+\mathrm{sinc}\ 2(f_c-2f_0)cT+\mathrm{sinc}\ 2(f_c+2f_0)cT}{4(a^2c^2+3abc+3b^2+a^2c^2\ \mathrm{sinc}\ 4f_ccT+2ac\sqrt{6(abc+b^2)}\ \mathrm{sinc}\ 2f_ccT)}]]$$

(5.35)

where $f_s = f_1+f_0$ and $f_d = f_1 - f_0$.

a. *Performance of the Suboptimum Receiver*

From Equation 3.8 we obtain

$$C = \frac{JSR\cdot SNR\cdot\varepsilon_s'}{1-JSR\cdot SNR\cdot\varepsilon_s'} \qquad (5.36)$$

where

$$JSR = \frac{\frac{c}{3}[\, a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT \; + 2ac\sqrt{6(abc+b^2)} \text{ sinc } 2f_c cT \,]}{E}.$$

$$SNR = E/(N_0/2)$$

and

$$\varepsilon_s{}' = 1 + \frac{(a^2c^2 + 3abc + 3b^2)(2\text{sinc } 2f_s cT - 2\text{sinc } 2f_d cT + \text{sinc } 4f_1 cT + \text{sinc } 4f_0 cT)}{2(a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT + 2ac\sqrt{6(abc+b^2)} \text{ sinc } 2f_c cT)}$$

$$+ a^2c^2[\frac{\text{sinc } 2(2f_c - f_s)cT + \text{sinc } 2(2f_c + f_s)cT - \text{sinc } 2(2f_c - f_d)cT - \text{sinc } 2(2f_c + f_d)cT}{2(a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT + 2ac\sqrt{6(abc+b^2)} \text{ sinc } 2f_c cT)}$$

$$+ \frac{\text{sinc } 4(f_c - f_1)cT + \text{sinc } 4(f_c + f_1)cT - \text{sinc } 4(f_c - f_0)cT + \text{sinc } 4(f_c + f_0)cT}{4(a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT + 2ac\sqrt{6(abc+b^2)} \text{ sinc } 2f_c cT)}]$$

$$+ 2ac\sqrt{6(abc+b^2)}[\frac{\text{sinc } 2(f_c - f_s)cT + \text{sinc } 2(f_c + f_s)cT - \text{sinc } 2(f_c - f_d)cT - \text{sinc } 2(f_c + f_d)cT}{2(a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{sinc} 4f_c cT + 2ac\sqrt{6(abc+b^2)} \text{sinc} 2f_c cT)}$$

$$+ \frac{\text{sinc } 2(f_c - 2f_1)cT + \text{sinc } 2(f_c + 2f_1)cT + \text{sinc } 2(f_c - 2f_0)cT + \text{sinc } 2(f_c + 2f_0)cT}{4(a^2c^2 + 3abc + 3b^2 + a^2c^2 \text{ sinc } 4f_c cT + 2ac\sqrt{6(abc-b^2)} \text{ sinc } 2f_c cT)}]. \tag{5.37}$$

Note that C in Equation 5.36 can be mininized by setting $\varepsilon_s{}'$ to its smallest possible value. Since with b = 0 and c = 1 $\varepsilon_s{}'$ equals 1.5 for $f_c = f_s/2$, 0.5 for $f_c = f_d/2$, and 0.75 for $f_c = f_1$ or $f_c = f_0$, it is apparent that the jammer is least damaging when f_c is set to $(f_1 - f_0)/2$, and most damaging when $f_c = (f_0 + f_1)/2$, the midpoint of the two operating frequencies. From Equation 3.7, we obtain actual receiver performance. Observe that for FSK

$$\frac{E(1-\rho)}{N_0} = \frac{SNR}{2} \qquad (5.38)$$

so that

$$P_{e,s} = \operatorname{erfc}\left(\sqrt{SNR\,\alpha/4}\right) \qquad (5.39)$$

where $\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_s')$. A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.10 with $\alpha_1 = 3\alpha_0$, and the parameters b set to zero, and c set to one.

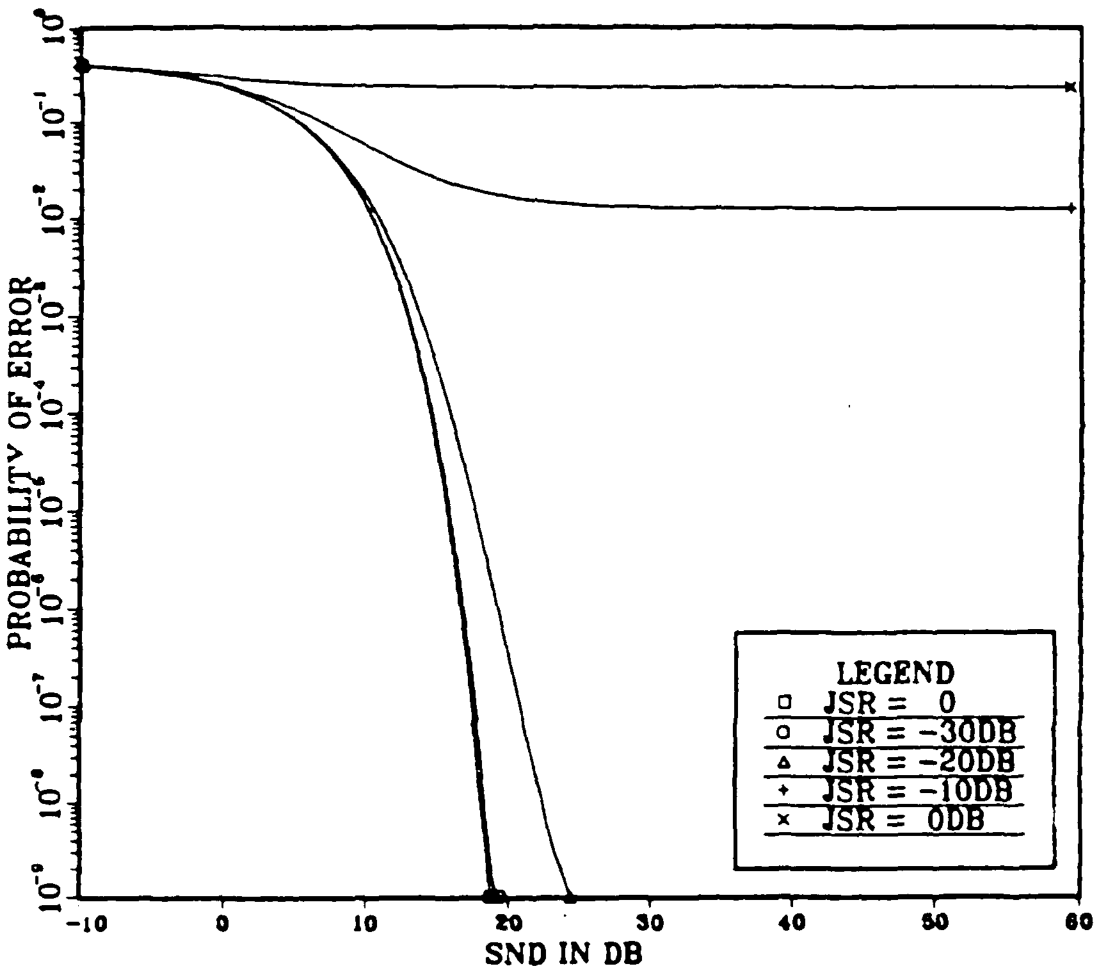


Figure 5.10 Performance of the Suboptimum Receiver for FSK Modulation with Sinusoidal Wave Interference.

b. Performance of the Optimum Receiver

The result of Equation 5.39 can once again be compared to the performance of the corresponding optimum receiver using Equation 4.5 and Equation 4.6. We thus obtain

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\left\{ \int_0^{cT} \frac{[\,2A\cos\pi(f_1+f_0)t\,\sin\pi(f_1-f_0)t\,]^2}{1+\left(\sqrt{\frac{2}{3}}\,ac\cos 2\pi f_c t+\sqrt{abc-b^2}\,\right)^2 \frac{N_0}{2}}\,dt \right.$$

$$\left. + \int_{cT}^{T} [\,2A\cos\pi(f_1+f_0)t\,\sin\pi(f_1-f_0)t\,]^2\,dt \right\}.$$

If set c to one and b to zero as a special case, then

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\int_0^{T} \frac{[\,2A\cos\pi(f_1+f_0)t\,\sin\pi(f_1-f_0)t\,]^2}{1+\left(\frac{a^2}{3}+\frac{a^2}{3}\cos 4\pi f_c t\right)\frac{N_0}{2}}\,dt$$

$$= \frac{A^2\eta}{4\pi f_c(a^2/3)}\int_0^{4\pi f_c T}\frac{[\,1+\cos(f_s x/2f_c)\,]\sin^2(f_d x/4f_c)}{1+\eta\cos x}\,dx \qquad (5.40)$$

where as before η = JSR·SNR/(1 + JSR·SNR), and $2f_cT$ is assumed to be an integer, say $2f_cT = n$. The integral of Equation 5.40 must be evaluated numerically or it can be approximated under certain circumstances. (The derivation of such an approximation is presented in Appendix B). If JSR = 0, q(t) = 0, then

$$\frac{E'(1-\rho')}{2} = \frac{SNR}{2} \qquad (5.41)$$

so that the actual receiver performance is obtained from

$$P_{e,o} = \begin{cases} \operatorname{erfc}\left(\sqrt{SNR/4}\right) & JSR = 0 \\ \\ \operatorname{erfc}\left(\sqrt{SNR\,\beta/4}\right) & JSR \neq 0 \end{cases} \qquad (5.42)$$

where

$$\beta = \frac{1}{2\pi f_c T(1 + JSR \cdot SNR)} \int_0^{4\pi f_c T} \frac{[\,1 + \cos(f_s y / 2f_c)\,] \, \sin^2(f_d y / 4f_c)}{1 + \eta \cos y} \, dy$$

$$\approx \frac{2(1 + JSR \cdot SNR) + 2\sqrt{1 - 2JSR \cdot SNR}}{2 + 5JSR \cdot SNR + 2(JSR \cdot SNR)^2 + (2 + 3JSR \cdot SNR)\sqrt{1 - 2JSR \cdot SNR}}.$$

A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.11 with parameters b set to zero, and c set to one.

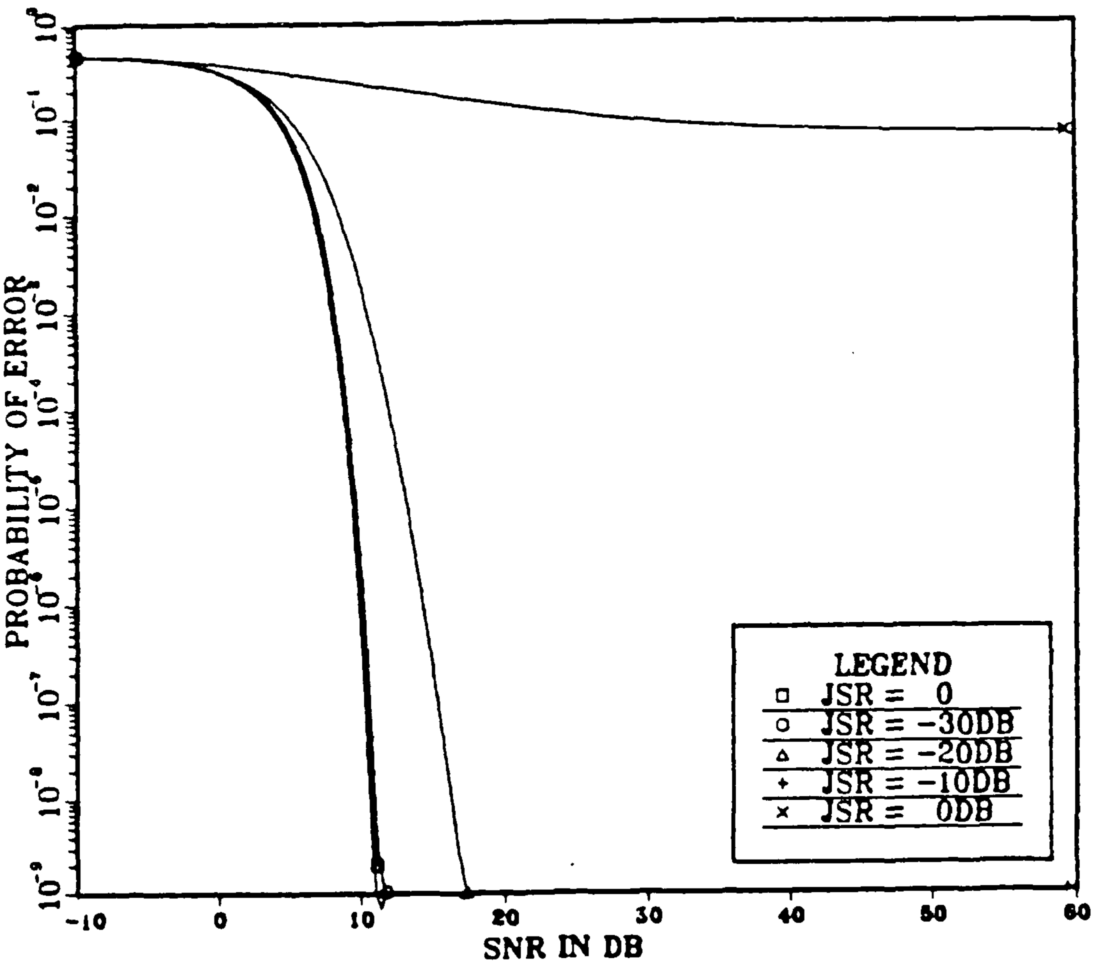


Figure 5.11 Performance of the Optimum Receiver for FSK Modulation with Sinusoidal Wave Interference.

It is not difficult to show that for $f_c = f_s/2$ and $c = 1$ is always exceeded by β. However, for increasing JSR·SNR, the ratio $\beta/\alpha \rightarrow 1.5$.

3. Comparison

Again, it becomes apparent that interference effects for FSK are not as severe as for PRK as the differences in performance between the optimum and suboptimum receiver for the former modulation scheme are not as large as those the latter modulation scheme.

C. PULSED WAVE

We define

$$q(t) = \begin{cases} \sqrt{\dfrac{2a^2c^2}{3} + 2abc + b^2} & \ell T_p \leq t \leq (\ell + \tfrac{1}{2})T_p \\ b & (\ell + \tfrac{1}{2})T_p \leq t \leq (\ell + 1)T_p \end{cases} \quad 0 \leq t \leq cT$$

$$\qquad\qquad 0 \qquad\qquad\qquad\qquad\qquad cT < t \leq 1 \qquad (5.43)$$

as yet another form of an interference generating function where $\ell = 0, 1, \ldots, p-1$, $pT_p = cT$, and $0 \leq c \leq 1$.

In this example, $q(t)$ is being pulsed many times between two levels in the interval $0 \leq t \leq cT$ as shown in Figure 5.12. From Equation 2.3

$$Q_a(0) = \frac{1}{T}\int_0^{cT} q^2(t)\,dt = \frac{1}{T}\sum_{\ell=0}^{p-1}\left(\frac{2a^2c^2}{3} + 2abc + b^2\right)\int_{\ell T_p}^{(\ell+\frac{1}{2})T_p} dt$$

$$+ \frac{1}{T}\sum_{\ell=0}^{p-1} b^2 \int_{(\ell+\frac{1}{2})T_p}^{(\ell+1)T_p} dt$$

$$= \frac{c}{3}(a^2c^2 + 3abc + 3b^2)$$

for the average power of the nonstationary interference.

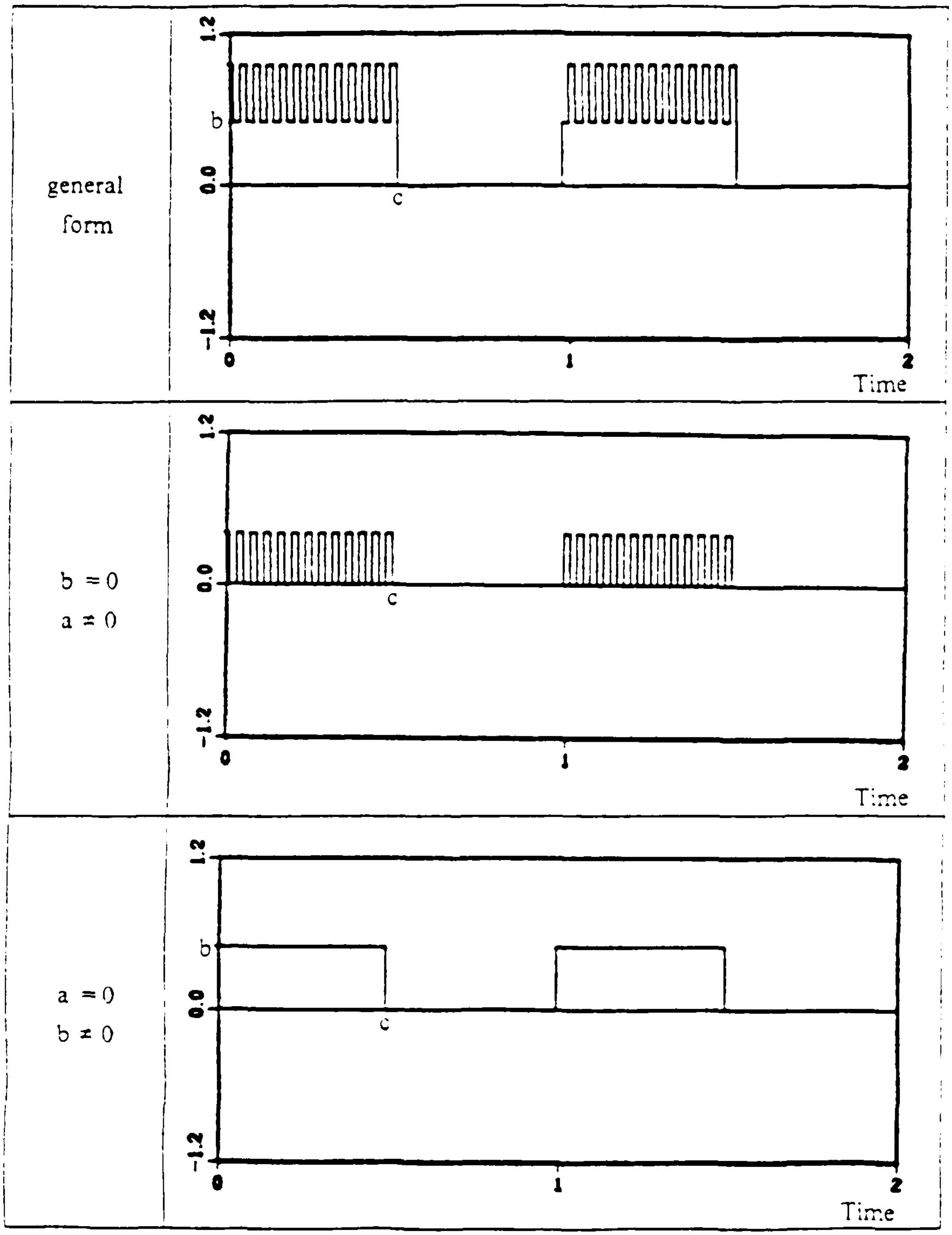


Figure 5.12 Several Variations of Pulsed Wave Interference as a Function of (normalized) Time.

1. **Phase Reversal Keying**

The following integral must first be evaluated in order to obtain $P_{e,s}$, namely

$$\int_T Q(t)\, s_d^2(t)\, dt = \int_0^{cT} q^2(t)\, [\, 4A^2(1 - m^2) \cos^2 2\pi f_r t\,]\, dt$$

$$= 2A^2T(1-m^2) \frac{c}{3} (a^2c^2 + 3abc + 3b^2) \cdot$$

$$\{1 + \frac{1}{4\pi f_r cT}[\sin 4\pi f_r cT + 2\sin^2 2\pi f_r cT \tan^{-1} (\pi f_r cT\ p)]\} \quad (5.44)$$

(See Appendix C for the pertinent derivations).

a. Performance of the Supoptimum Receiver

The pulsed wave interference of Equation 5.43 is now used to determine the receiver probability of error for PRK focusing first on the performance of the receiver of Figure 1.1. From Equation 3.8 and Equation 5.44, we obtain

$$C = \frac{JSR \cdot SNR \cdot \varepsilon_p}{1 + JSR \cdot SNR \cdot \varepsilon_p} \quad (5.45)$$

where

$$JSR = [\, \frac{c}{3} (a^2c^2 + 3abc + 3b^2)\,]\, E$$

$$SNR = \frac{E}{N_0\ 2}$$

and

$$\varepsilon_p = \{1 + \frac{1}{4\pi f_r cT} [\, \sin 4\pi f_r cT + 2\sin^2 2\pi f_r cT \tan (\pi f_r cT\ p)\,]\, \}.$$

Actual receiver performance (in terms of probability of error) for the conventional receiver is obtained from Equation 3.7. Observe that for PRK,

$$\frac{E(1-\rho)}{N_0} = \frac{A^2T}{N_0}(1-m^2) = SNR(1-m^2) \tag{5.46}$$

and

$$1 - C = \frac{1}{1 + JSR \cdot SNR \cdot \varepsilon_p}$$

so that

$$P_{e,s} = \operatorname{erfc}\left[\sqrt{SNR(1-m^2)\alpha/2}\,\right] \tag{5.47}$$

where

$$\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_p).$$

A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.13 with m set to zero, and the parameters b set to zero, and c set to one.

b. Performance of the Optimum Receiver

The previous result for $P_{e,s}$ can be compared to the performance of the corresponding optimum receiver. From Equation 4.5 and Equation 4.6, we obtain

$$\frac{E'(1-\rho')}{2} = \frac{4A^2(1-m^2)}{2N_0}\left[\int_0^{cT}\frac{\cos^2 2\pi f_r t}{1 + q^2(t)\frac{N_0}{2}}\,dt + \int_{cT}^{T}\cos^2 2\pi f_r t\,dt\right].$$

When b is set to zero and c set to one as a special case, then $q(t) = \sqrt{2a^2/3}$, so that

$$\frac{E'(1-\rho')}{2} = \frac{A^2T(1-m^2)}{N_0}\left(\frac{1+\frac{\beta^2}{3}\frac{N_0}{2}}{1+\frac{2\beta^2}{3}\frac{N_0}{2}}\right)$$

$$= SNR(1-m^2)\left(\frac{1+JSR\cdot SNR}{1+2JSR\cdot SNR}\right).$$

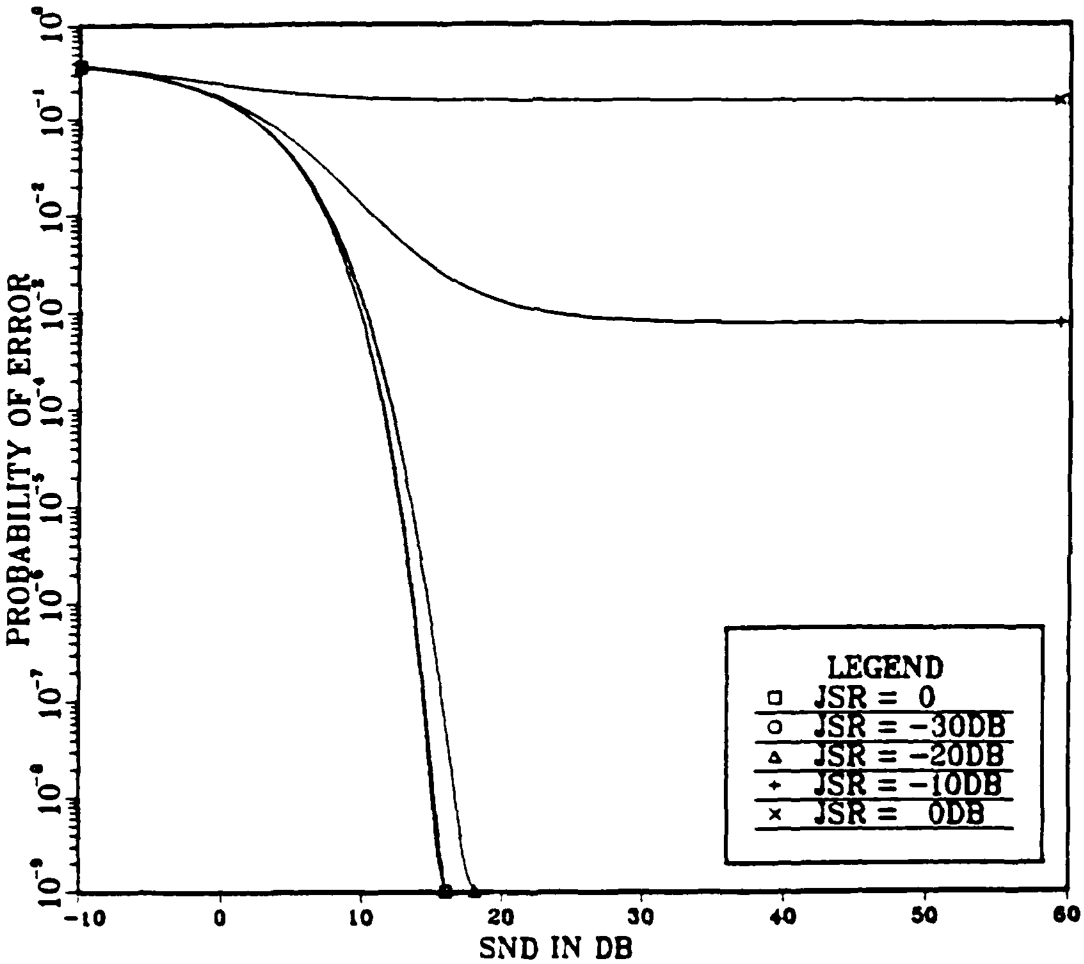


Figure 5.13 Performance of the Suboptimum Receiver of PSK Modulation with the Pulsed Wave Interference.

In the case of JSR = 0, q(t) = 0, so that directly from Equation 4.6 we have

$$\frac{E'(1-\rho')}{2} = \frac{1}{2N_0}\int_T s_d^2(t)\,dt = \frac{E(1-\rho)}{N_0} = SNR\,(1-m^2) \qquad (5.48)$$

so that

$$P_{e,o} = \begin{cases} \operatorname{erfc}\left[\sqrt{SNR(1-m^2)/2}\right] & JSR = 0 \\ \operatorname{erfc}\left[\sqrt{SNR(1-m^2)\beta/2}\right] & JSR \neq 0 \end{cases} \tag{5.49}$$

where

$$\beta = \frac{1 + JSR \cdot SNR}{1 + 2\,JSR \cdot SNR}.$$

A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.14 with m set to zero, and parameters b set to zero, with c set to one.

From previous results we observe that $P_{e,o} \leq P_{e,s}$ as

$$\alpha = \frac{1}{1 + JSR \cdot SNR} < \frac{1 + JSR \cdot SNR}{1 + 2JSR \cdot SNR} = \beta$$

where c = 1 has been used in the above expression for α. Note that the above inequality becomes stronger as JSR·SNR increases.

2. Frequency Shift Keying

In order to obtain $P_{e,s}$ for FSK modulation, the following integral must evaluated, namely

$$\int_T Q(t)\, s_d^2(t)\, dt = \sum_{\ell=0}^{p-1} \left(\frac{2a^2c^2}{3} + 2abc + 2b^2\right) \int_{\ell T_p}^{(\ell+1/2)T_p} [4A^2 \cos^2 \pi(f_1+f_0)t \sin^2 \pi(f_1-f_0)t]\, dt$$

$$= \frac{2A^2}{3}(a^2c^2 + 3abc + 3b^2) \sum_{\ell=0}^{p-1} \int_{\ell T_p}^{(\ell+1/2)T_p} \left(1 + \cos\frac{2\pi f_s T t}{T} - \cos\frac{2\pi f_d T t}{T} - \frac{1}{2}\cos\frac{4\pi f_1 T t}{T} - \frac{1}{2}\cos\frac{4\pi f_0 T t}{T}\right) dt.$$

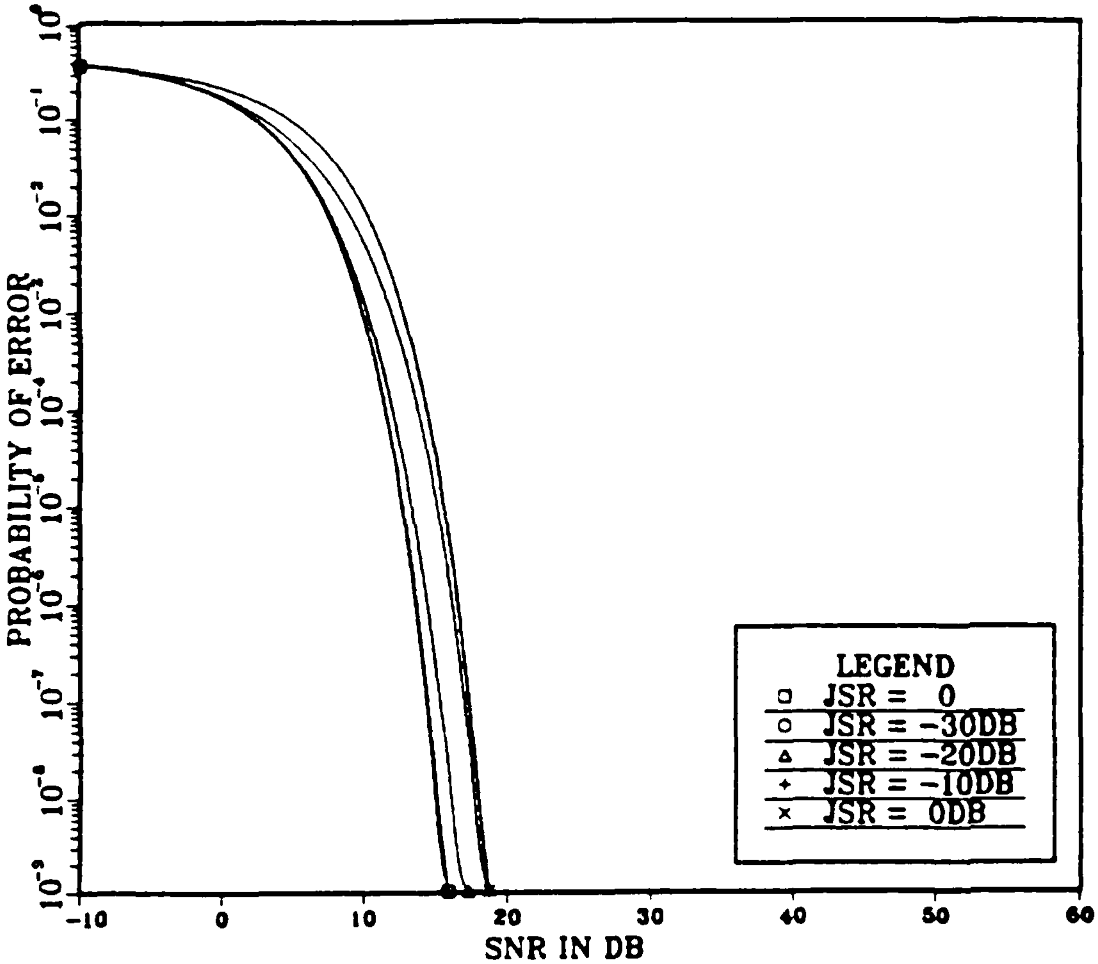


Figure 5.14 Performance of the Optimum Receiver for PSK Modulation with Pulsed Wave Interference.

All integrals of cosine terms in the previous expression are of the form

$$\int_{\ell T_p}^{(\ell+\frac{1}{2})T_p} \cos\frac{\pi k t}{T}\, dt = \left[\frac{\sin \pi k t/T}{\pi k/T}\right]_{\ell T_p}^{(\ell+\frac{1}{2})T_p}$$

$$= \frac{\sin \pi k c(\ell+\frac{1}{2})p - \sin \pi k c \ell p}{\pi k/T}$$

where $k = 2f_i T$. Therefore The sum

$$\sum_{\ell=0}^{p-1} [\, \sin \pi kc(\ell + ½)/p - \sin (\pi kc\ell/p) \,] \tag{5.50}$$

$$= \sin^2 \frac{\pi kc}{2} \tan \frac{\pi kc}{4p} + 2\sin \pi kc$$

(See Appendix D for the pertinent derivations). Therefore

$$\int_T Q(t)\, s_d^2(t)\, dt = \frac{A^2Tc}{3}(a^2c^2 + 3abc + 3b^2)\cdot$$

$$\{ 1 + \frac{1}{\pi(f_1 + f_0)cT} [\, \sin^2 \pi(f_1 + f_0)cT \tan \frac{\pi(f_1 + f_0)cT}{2p} + 2\sin 2\pi(f_1 + f_0)cT \,]$$

$$- \frac{1}{\pi(f_1 - f_0)cT} [\, \sin^2 \pi(f_1 - f_0)cT \tan \frac{\pi(f_1 - f_0)cT}{2p} + 2\sin 2\pi(f_1 - f_0)cT \,]$$

$$- \frac{1}{2\pi f_1 cT} (\, \sin^2 2\pi f_1 cT \tan \frac{\pi f_1 cT}{p} + 2\sin 4\pi f_1 cT \,)$$

$$- \frac{1}{2\pi f_0 cT} (\, \sin^2 2\pi f_0 cT \tan \frac{\pi f_0 cT}{p} + 2\sin 4\pi f_0 cT \,) \}. \tag{5.51}$$

a. Performance of the Suboptimum Receiver

From Equation 3.8 and Equation 5.51, we obtain

$$C = \frac{JSR\cdot SNR\cdot \varepsilon_p'}{1 + JSR\cdot SNR\cdot \varepsilon_p'} \tag{5.52}$$

where

$$JSR = [\, \frac{c}{3} (a^2c^2 + 3abc + 3b^2) \,] / E$$

$$SNR = \frac{E}{N_0/2}$$

and

$$\begin{aligned}\varepsilon_p' = \; & 1 + \frac{1}{\pi(f_1+f_0)cT}\left[\sin^2 \pi(f_1+f_0)cT \tan \frac{\pi(f_1+f_0)cT}{2p} + 2\sin 2\pi(f_1+f_0)cT \right] \\ & - \frac{1}{\pi(f_1-f_0)cT}\left[\sin^2 \pi(f_1-f_0)cT \tan \frac{\pi(f_1-f_0)cT}{2p} + 2\sin 2\pi(f_1-f_0)cT \right] \\ & - \frac{1}{2\pi f_1 cT}\left(\sin^2 2\pi f_1 cT \tan \frac{\pi f_1 cT}{p} + 2\sin 4\pi f_1 cT \right) \\ & - \frac{1}{2\pi f_0 cT}\left(\sin^2 2\pi f_0 cT \tan \frac{\pi f_0 cT}{p} + 2\sin 4\pi f_0 cT \right). \end{aligned} \tag{5.53}$$

Actual receiver performance (in terms of probability of error) is obtained from Equation 3.7. Observe for FSK modulation

$$\frac{E(1-\rho)}{N_0} = \frac{SNR}{2} \tag{5.54}$$

and

$$1 - C = \frac{1}{1 + JSR \cdot SNR \cdot \varepsilon_p'}$$

so that

$$P_{e,s} = \operatorname{erfc}\left(\sqrt{SNR\, \alpha / 4} \right) \tag{5.55}$$

where

$$\alpha = 1/(1 + JSR \cdot SNR \cdot \varepsilon_p').$$

A plot of $P_{e,s}$ as a function of SNR and JSR is shown in Figure 5.15 with parameters b set to zero, and c set to one.

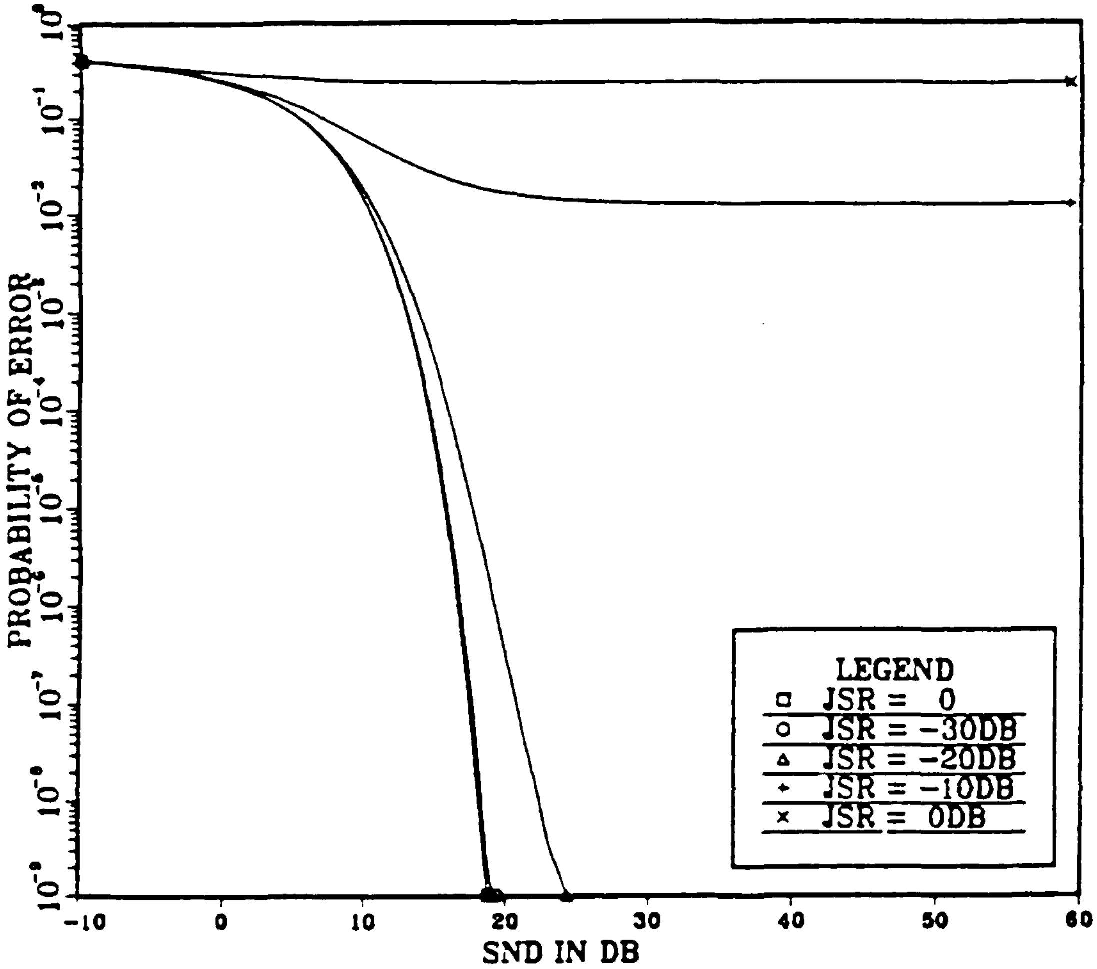


Figure 5.15 Performance of the Suboptimum Receiver for FSK Modulation with Pulsed Wave Interference.

b. Performance of the Optimum Receiver

The above result for the probability of error of the suboptimum receiver can be compared to the performance of the optimum receiver using Equation 4.5 and Equation 4.6. For the FSK modulation, the optimum receiver performance is obtained by first evaluating

$$\frac{E(1-\rho)}{2} = \frac{1}{2N_0}\left[\int_0^{cT} \frac{s_d^2(t)}{1 + q^2(t)\frac{N_0}{2}}\,dt + \int_{cT}^{T} s_d^2(t)\,dt\right].$$

If we let b = 0 and c = 1 as a special case, then $Q_a(0) = a^2/3$. Thus we can simplify the above equation to obtain

$$\frac{E'(1 - \rho')}{2} = \frac{A^2T}{2N_0}\left(\frac{1 + \dfrac{a^2}{3}\,\dfrac{N_0}{2}}{1 + \dfrac{2a^2}{3}\,\dfrac{N_0}{2}} \right) = \frac{SNR}{2}\left(\frac{1 + JSR\cdot SNR}{1 + 2JSR\cdot SNR} \right). \tag{5.56}$$

When JSR = 0, q(t) = 0, thus

$$\frac{E'(1 - \rho')}{N_0} = \frac{SNR}{2} \tag{5.57}$$

so that the actual performance being obtained from

$$P_{e,o} = \begin{cases} \mathrm{erfc}\left(\sqrt{SNR/4}\right) & JSR = 0 \\ \\ \mathrm{erfc}\left(\sqrt{SNR\,\beta/4}\right) & JSR \neq 0 \end{cases} \tag{5.58}$$

where

$$\beta = \frac{1 + JSR\cdot SNR}{1 + 2JSR\cdot SNR}.$$

A plot of $P_{e,o}$ as a function of SNR and JSR is shown in Figure 5.16 with parameters b set to zero, and c set to one.

3. Comparison

We observe here identical results on suboptimality for FSK as for PRK. That is, since α and β are similar in form for both modulation schemes, we find here that FSK modulation is no less susceptible to jamming than PSK modulation when using a suboptimum receiver to process the signal, the noise, and the jamming.

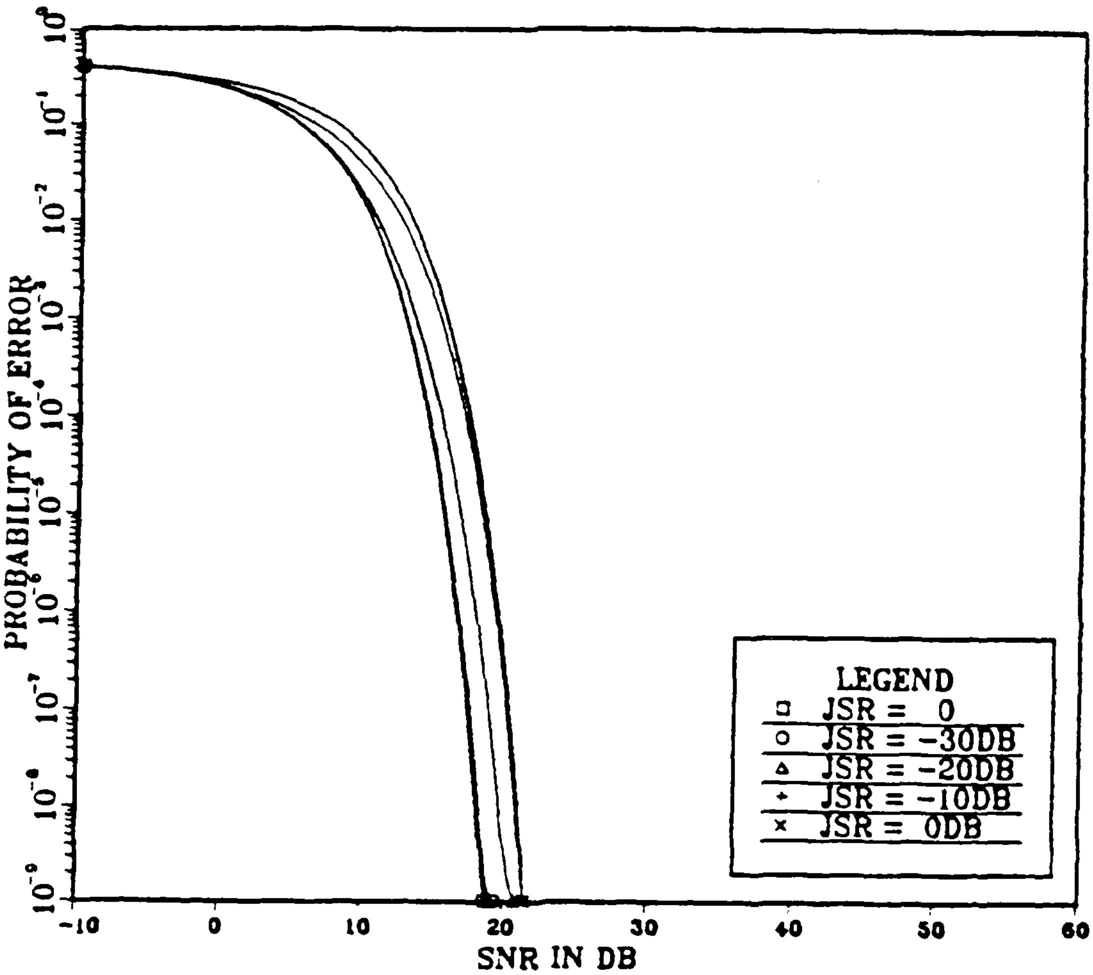


Figure 5.16 Performance of the Optimum Receiver for FSK Modulation with Pulsed Wave Interference.

VI. CONCLUSIONS

In this thesis general results have been obtained for the performance of optimum and suboptimum (conditional) digital communication receivers operating in the presence of (stationary) AWGN and nonstationary AWGN generated via a specific model. The general results show that the suboptimum receivers have performance (i.e., probability of error, $P_{e,s}$) that is always inferior to that of their optimum counterparts (expressed also as probability of error, $P_{e,o}$).

The many examples worked out in Chapter V for PRK and FSK modulation for different nonstationary AWGN interference clearly demonstrate the level of the suboptimality of conditional receivers. Perhaps more importantly, the examples demonstrate the high degree of vulnerability of conventional receivers to jamming, having characteristics similar to the nonstationary AWGN interference a model used. The $P_{e,s}$ plots demonstrate that without powerful jamming, that is for relatively low JSR values, the receivers can be rendered almost completely ineffective as their error probabilities are significantly higher than the 10^{-3} BEP military standard.

The optimum receivers proposed however perform significantly better when compared to the suboptimum receivers and in fact in all cases, with sufficient SNR, are able to overcome the effect of jamming. It is important to note however that these receivers, in order to be optimum most have knowledge about the nonstationary (jamming) interference that would normally not be available. Consequently the actual performance of these receivers may in practice be work worse than predicted and perhaps worse than the suboptimum receivers.

APPENDIX A

DERIVATION OF THE APPROXIMATION TO EQ. 5.10

From Equation 5.10 we must evaluate

$$\frac{E'(1-\rho')}{2} = \frac{SNR(1-m^2)}{\sqrt{3JSR \cdot SNR}} \left[\tan^{-1}\sqrt{3JSR \cdot SNR} + \int_0^{\sqrt{3JSR \cdot SNR}} \frac{\cos(4\pi f_r T x \sqrt{3JSR \cdot SNR})}{1+x^2} dx \right].$$

If we assume that $\sqrt{3JSR \cdot SNR} >> 1$, then

$$\frac{E'(1-\rho')}{2} \approx \frac{SNR(1-m^2)}{\sqrt{3JSR \cdot SNR}} \left[\tan^{-1}\sqrt{3JSR \cdot SNR} + \frac{\pi}{2}\, e^{\left(\frac{-4\pi f_r T}{\sqrt{3JSR \cdot SNR}}\right)} \right]$$

as the above integral can be evaluated in closed form when $\sqrt{JSR \cdot SNR} >> 1$.

If we further assume that JSR·SNR is so large that the exponential is nearly unity and $\tan^{-1}\sqrt{3JSR \cdot SNR} \approx \pi/2$, then

$$\frac{E'(1-\rho')}{2} \approx \frac{SNR(1-m^2)\pi}{\sqrt{3JSR \cdot SNR}}.$$

APPENDIX B

DERIVATION OF THE APPROXIMATION TO EQ. 5.40

Recall from the discussion that for the suboptimum receiver for FSK modulated signal, placing f_c at f_s 2 in the sinusoidal wave interference frequency is most damaging to the performance of the receiver. For this reason, f_c is set to f_s 2 and Equation 5.40 is approximated under this constraint. We obtain

$$\frac{E(1-\rho')}{2} = \frac{A^2T\eta}{n(a^2\ 3)}\ \frac{1}{2\pi}\int_0^{2\pi n} \frac{(1+\cos x)\sin^2(f_d x\ 2f_s)}{1+\eta\cos x}\,dx$$

where η was defined before.

Observe that typically $f_d\ 2f_s << 1$. (For $f_1 = 3.002$ MHz and $f_0 = 3$ MHz, $f_d\ 2f_s - 1.67 \times 10^{-4}$). Since $(1 + \cos x)\ (1 + \eta\cos x)$ is periodic in 2π, and $\sin^2(f_d\ 2f_s)$ is essentially constant for $0 \le x \le 2\pi$, we obtain

$$\frac{E(1-\rho')}{2} \approx \frac{A^2T\eta}{n(a^2\ 3)} \sum_{k-1}^{n} \left[\frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos x}{1+\eta\cos x}\,dx \quad \sin^2\left(\frac{f_d}{2f_s}\,2\pi k\right)\right]$$

$$= \frac{A^2T\eta}{n(a^2\ 3)}\left[(1-p)\frac{1+p^2}{1-p^2}\right] \sum_{p-1}^{n} \sin^2\left(\frac{\pi f_d k}{2f_s}\right)$$

and p is given by Equation 5.30. The finite sum can be evaluated in closed form and we finally obtain

$$\frac{E(1-\rho')}{2} \approx \frac{(A^2T\ 2)\eta}{(a^2\ 3)}\left(\frac{1+p^2}{1-p^2}\right)\left\{1 - \frac{[\cos(n+1)(\pi f_d\ f_s)]\sin(n\pi f_d\ f_s)}{n\sin(\pi f_d\ f_s)}\right\}$$

$$= \frac{1}{JSR}\left[\frac{2\eta + 2\eta\sqrt{1-\eta^2}}{2+\eta-\eta 2+(2+\eta)\sqrt{1-\eta^2}}\right]\left\{1 - [\cos(n+1)(\pi f_d\ f_s)]\frac{\sin(n\pi f_d\ f_s)}{(n\pi f_d\ f_s)}\right\}.$$

For n large the term

$$[\cos(n+1)(\pi f_d f_s)] \frac{\sin(n\pi f_d f_s)}{n\pi f_d f_s} \approx 0,$$

thus we obtain the approximation

$$\frac{E(1-\rho')}{2} \approx \left(\frac{SNR}{2}\right)\cdot$$

$$\left[\frac{2(1 + JSR\cdot SNR) + 2\sqrt{1 + 2JSR\cdot SNR}}{2 + 5JSR\cdot SNR + 2(JSR\cdot SNR)^2 + (2 + 3JSR\cdot SNR)\sqrt{1 + 2JSR\cdot SNR}} \right].$$

APPENDIX C

DERIVATION OF EQUATION 5.44

For PRK modulation with pulsed wave interference, from Equation 5.44 we evaluate

$$\int_T Q(t)\, s_d^2(t)\, dt = \int_0^{cT} q^2(t)\,[\,4A^2(1 - m^2)\cos^2 2\pi f_r t\,]\, dt$$

$$= 2A^2(1 - m^2)\left(\frac{2a^2c^2}{3} + 2abc + 2b^2\right) \sum_{\ell=0}^{p-1} \int_{\ell T_p}^{(\ell+\frac{1}{2})T_p} (1 + \cos 4\pi f_r t)\, dt$$

$$= 4A^2(1 - m^2)\left(\frac{a^2c^2}{3} + abc + b^2\right) \sum_{\ell=0}^{p-1} \left[\, t + \frac{\sin 4\pi f_r t}{4\pi f_r}\,\right]_{\ell T_p}^{(\ell+\frac{1}{2})T_p}$$

$$= 2A^2(1 - m^2)\left(\frac{a^2c^2}{3} + abc + b^2\right) \sum_{\ell=0}^{p-1} \{T_p + [\sin 4\pi f_r(\ell+\tfrac{1}{2})T_p - \sin 4\pi f_r \ell T_p]\, 2\pi f_r\}.$$

Thus, for the sum term we have

$$\sum_{\ell=0}^{p-1} [\,\sin 4\pi f_r(\ell+\tfrac{1}{2})T_p - \sin 4\pi f_r \ell T_p\,]$$

$$= \sin 2\pi f_r T_p \sum_{\ell=0}^{p-1} \cos 4\pi f_r \ell T_p - (1 - \cos 2\pi f_r T_p) \sum_{\ell=0}^{p-1} \sin 4\pi f_r \ell T_p$$

$$= \sin 2\pi f_r T_p \cos 2\pi f_r (p\text{-}1) T_p \sin 2\pi f_r p T_p \sin 2\pi f_r T_p$$

$$- 2\sin^2 \pi f_r T_p \sin 2\pi f_r (p\text{-}1) T_p \sin 2\pi f_r p T_p \sin 2\pi f_r T_p$$

since

$$\sum_{\ell=0}^{k} \cos 2\pi f \ell x = \cos k 2\pi f x \sin 2\pi f (k\text{-}1) x \sin 2\pi f x$$

and

$$\sum_{\ell=0}^{k} \sin 2\pi f\ell x = \sin k2\pi fx \sin 2\pi f(k-1)x \sin 2\pi fx.$$

Thus the sum term

$$\sum_{\ell=0}^{p-1} [\, \sin 4\pi f_r(\ell + ½)T_p - \sin 4\pi f_r \ell T_p \,]$$

$$= \cos (p-1)2\pi f_r T_p \sin 2\pi f_r T_p - \tan \pi f_r T_p \sin (p-1)2\pi f_r T_p \sin 2\pi f_r pT_p$$

$$= (\, \cos 2\pi f_r pT_p \cos 2\pi f_r T_p + \sin 2\pi f_r pT_p \sin 2\pi f_r T_p) \sin 2\pi f_r pT_p$$

$$- \tan \pi f_r T_p \sin 2\pi f_r pT_p (\sin 2\pi f_r pT_p \cos 2\pi f_r T_p - \cos 2\pi f_r pT_p \sin 2\pi f_r T_p)$$

$$= (\sin 4\pi f_r pT_p \cos 2\pi f_r T_p)\ 2 + \sin^2 2\pi f_r pT_p \sin 2\pi f_r T_p$$

$$- \sin^2 2\pi f_r pT_p \cos 2\pi f_r T_p \tan \pi f_r T_p + (\sin 4\pi f_r pT_p \sin 2\pi f_r T_p \tan \pi f_r T_p)\ 2$$

$$= (\sin 4\pi f_r cT \cos 2\pi f_r cT\ p)\ 2 + \sin^2 2\pi f_r cT \sin 2\pi f_r cT\ p$$

$$- \sin^2 2\pi f_r cT \cos 2\pi f_r cT\ p \tan \pi f_r cT\ p + (\sin 4\pi f_r cT \sin 2\pi f_r cT\ p \tan \pi f_r cT\ p)\ 2$$

where $pT_p = cT$. Since $\sin 2\pi f_r cT\ p \tan \pi f_r cT\ p = 2\sin^2 \pi f_r cT\ p$,

$$\sum_{\ell=0}^{p-1} [\, \sin 4\pi f_r(\ell + ½)T_p - \sin 4\pi f_r \ell T_p \,]$$

$$= [\, \sin 4\pi f_r cT (\cos 2\pi f_r cT\ p + 2\sin^2 2\pi f_r cT\ p)\,]\ 2$$

$$+ \sin^2 2\pi f_r cT\ (\sin 2\pi f_r cT\ p - \cos 2\pi f_r cT\ p \tan \pi f_r cT\ p).$$

Observe that the terms in the first parenthesis is unity and in the second parenthesis is $\tan \pi f_r cT\, p$, thus

$$\sum_{\ell=0}^{p-1} [\, \sin 4\pi f_r(\ell + 1/2)T_p - \sin 4\pi f_r \ell T_p \,]$$

$$= \sin^2 2\pi f_r T \tan \pi f_r cT\, p + (\sin 4\pi f_r cT)\, 2.$$

The remaining sum term is

$$\sum_{\ell=0}^{p-1} (\, T_p + 1/2\pi f_r) = cT(1 + 1/2\pi f_r cT).$$

Thus from above results we finally obtain

$$\int_T q^2(t)s_d^2(t)dt = 2A^2T(1-m^2)\frac{c}{3}(a^2c^2 + 3abc + 3b^2)\cdot$$

$$[\, 1 + \frac{1}{4\pi frcT}(\sin 4\pi f_r cT + 2\sin^2 2\pi f_r cT \tan \pi f_r cT\, p\,)\,].$$

APPENDIX D

DERIVATION OF EQUATION 5.50

The sum term in Equation 5.50, namely

$$\sum_{\ell=0}^{p-1} [\, \sin \pi kc(\ell + ½)/p - \sin \pi kc\ell/p \,]$$

yields

$$= \sum_{\ell=0}^{p-1} [\, \sin \pi kc\ell/p \cos \pi kc/2p + \cos \pi kc\ell/p \sin \pi kc/2p - \sin \pi kc\ell/p \,]$$

$$= \sin \pi kc/2p \sum_{\ell=0}^{p-1} \cos \pi kc\ell/p - (1 - \cos \pi kc/2p) \sum_{\ell=0}^{p-1} \sin \pi kc\ell/p$$

$$= [\, \sin \pi kc/2p \cos (p\text{-}1)\pi kc/2p \sin \pi kcp/2p \,] \sin \pi kc/2p$$

$$- [\, 2\sin^2 \pi kc/4p \sin (p\text{-}1)\pi kc/2p \sin \pi kcp/2p \,] \sin \pi kc/2p$$

$$= \cos (p\text{-}1)\pi kc/2p \sin \pi kc/2 - \tan \pi kc/4p \sin (p\text{-}1) \pi kc/2p \sin \pi kc/2$$

$$= \sin \pi kc/2 \,[\cos (p\text{-}1)\pi kc/2p - \tan \pi kc/4p \sin (p\text{-}1)\pi kc/2p \,]$$

$$= \sin \pi kc/2 \,[\, \sin \pi kc/2 (\sin \pi kc/2p - \tan \pi kc/4p \cos \pi kc/2p)$$

$$+ \cos \pi kc/2 (\cos \pi kc/2p + \tan \pi kc/4p \sin \pi kc/2p) \,]$$

$$= \sin \pi kc/2 (\sin \pi kc/2 \tan \pi kc/4p + \cos \pi kc/2)$$

$$= \sin^2 \pi kc/2 \tan \pi kc/4p + ½\sin \pi kc.$$

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